Midterm Examination

This is a 2 hour exam. The exam is closed book and closed notes except for one sheet (two sides) of notes. Begin each problem on a new page. Justify all your computations, partial credit will be given for answers that are well reasoned even if the final result is wrong.

1. Short questions.
   a. Assume that you are able to sample a random variable $X$ from an exponential distribution with parameter 1 ($f_X(x) = e^{-x}$ for $x \geq 0$). Explain how to generate:
      i. a Bernouilli random variable $Y$ with parameter $\frac{1}{e^2}$,
      ii. a random variable $Z$ uniformly distributed between 1 and 3.
   b. You bump into a student in the Main Quad. If the average age of Stanford students is 20 years, bound the probability that the student is over 30 years old.
   c. Let $X$ and $Z$ be uncorrelated random variables with zero mean and variance $N$. Set $Y_1 = Z$ and $Y_2 = XZ$.
      What is the MMSE estimate of $X$ given $Y_1$ and $Y_2$ and its MSE?

2. Big Game. Stanford hires you to estimate whether it will rain during the Big Game. Checking past data you determine that the chance of rain is of 20%. You model this with the random variable $R$ with pmf
   \[ p_R(1) = 0.2, \quad p_R(0) = 0.8, \]
   where $R = 1$ means that it rains and $R = 0$ that it doesn’t. Your first idea is to be lazy and just use the forecast of a certain website. Analyzing data from previous forecasts, you determine that this website is right 70% of the time. You model this with a random variable $W$ that satisfies
   \[ P(W = R) = 0.7, \quad P(W \neq R) = 0.3. \]
   a. What is your prediction given the forecast of the website (use the MAP estimate of $R$ given $W$)? What is the probability of error under your model?
   b. Is it more reasonable to assume that $H$ and $W$ are independent, or that they are conditionally independent given $R$? Explain why.
      You assume that $H$ and $W$ are conditionally independent given $R$. More research establishes that conditioned on $R = 1$ $H$ is uniformly distributed between 0.5 and 0.7, whereas conditioned on $R = 0$ $H$ is uniformly distributed between 0.1 and 0.6.
   c. What is your forecast if $H = 0.65$ and the website predicts no rain?
d. What is your forecast if $H = 0.55$ and the website predicts rain?

e. What is the probability of error under this new model?

3. Short questions.

a. Let $X_1, X_2, \ldots, X_n$ be independent random variables uniformly distributed on $[0, 1]$. Let $Z = \max(X_1, X_2, \ldots, X_n)$. Find the pdf of $Z$.

b. Let $Z$ be a two dimensional vector uniformly distributed over the unit circle. Let $X$ be the $x$-coordinate of $Z$. What is the variance of $X$?

c. You are calling your friends to see if there is a lecture on Wednesday. Each friend you call knows the answer with probability $q$ independent of everyone else. You keep calling your friends until you learn the answer (i.e. until you find a friend who knows the answer). Each time you call, you engage in conversation whose duration is an exponential random variable with parameter $\lambda$, independent of everything else. Find the mean and the variance of the total time you spent calling your friends. Hint: There is a table at the end of the midterm listing the mean and variance of famous r.v.s which you may find useful.

d. i. Let $X$ and $Y$ be two random variables such that $f_{X|Y}(x|y) = \text{unif}[-1, 1]$. What is the MMSE estimate of $X$ given $Y$ and the corresponding MSE?

ii. Give an example of two random variables $X$ and $Y$ which are not independent but the MMSE when estimating $X$ from $Y$ is equal to $\text{Var}(X)$. Hint: you may want to start with the example in part (i) and modify it.

e. Give an example of two random variables $X$ and $Y$ that are not independent but uncorrelated and both $X$ and $Y$ have non-zero mean.

f. Prove that for any random variable $X$ and a real number $t \geq 0$,

$$P(X > a) \leq e^{-(at-\ln\mathbb{E}[e^{tX}])}.$$  

This is called Chernoff’s bound. It is usually tighter than Markov’s and Chebyshev’s bounds.