Q. 5: \[ LR(y) = \frac{f_{Y/X}(y | X=1)}{f_{Y/X}(y | X=0)} \] if \[ Y = hX + W \] 

Q. 6. \[ Y = AX + W \quad W \sim N(0, \sigma) \]
a) \[ \hat{X}(y, a) \]

\[ LR(y \mid a) = \frac{f_{Y,A}(y, a \mid X=1)}{f_{Y,A}(y, a \mid X=1)} \] if \[ X \leq 1 \]
Scenario 1: \( Y, A \) observed, \( A \geq 0 \Rightarrow \hat{X} = 1 \)

Scenario 2: \( Y \) observed, \( A \geq 0 \Rightarrow \hat{X} = -1 \)

Scenario 3: \( Y \) observed, \( \text{pdf} A \) is sym around 0

\[ f_{Y,A}(y,a \mid X = 1) \]

\[ = f_{A \mid X}(a \mid X = 1) \cdot f_{Y \mid X,A}(y \mid A=a, X = 1) \]

\[ = \frac{f_{A}(a) \cdot f_{Y \mid X,A}(y \mid a,x=1)}{f_{A}(a) \cdot f_{Y \mid X,A}(y \mid a,x=0)} \leq \]

\[ \frac{f_{A}(a) \cdot f_{Y \mid X,A}(y \mid a,x=0)}{f_{A}(a) \cdot f_{Y \mid X,A}(y \mid a,x=0)} \]

Estimation

\[ X \xrightarrow{\text{channel}} Y \xrightarrow{\text{estimate}} \hat{X} \]

\( X \) is continuous

\[ \mathbb{P}_Y (\hat{X} \neq X) \leq \lambda (x, \hat{x}) \]

\[ C(x, \hat{x}) = \text{cost function (loss function)} \]

Criterion: \( E [ C(X, \hat{X}) ] \)
\[ c(x, \hat{x}) = |x - \hat{x}| \]

\[ c(x, \hat{x}) = (x - \hat{x})^2 \]

\[ c(x, \hat{x}) = \begin{cases} 1 & \text{if } x \neq \hat{x} \\ 0 & \text{else} \end{cases} \rightarrow \text{error probability} \]

Continuity, but not smooth.

Continuous and smooth.

not continuous

\[ 2 - \text{square error} \]

Estimator: MMSE = minimum mean square estimator

\[ \min_\hat{X} \mathbb{E} \left[ c(X, \hat{X}(Y)) \right] \]

Extension to vector

\[ c(x, \hat{x}) = \| x - \hat{x} \|^2 = \sum (x_i - \hat{x}_i)^2 \]

Focus on scalar case first.
Simple solution to NMSE estimation

\[ X \rightarrow \text{channel} \rightarrow Y \]

\[ \frac{f_X}{f_Y|x} \]

\[ \hat{X}^* = \text{MMSE} \]

\[ \min_{\hat{X}} \mathbb{E} \left[ (X - \hat{X}(Y))^2 \right] \]

Condition \( Y \)

\[ \Pr \{ \hat{X}(Y) \neq X \} \]

Analogous to optimal design: NAP rule.

\[ \min_{\hat{X}} \int \left[ \mathbb{E} \left[ (X - \hat{X}(Y))^2 \right] \bigg| Y = y \right] f_Y(y) \, dy. \]

(totally prob rule applied to expectation, conditioning \( W \) continues)

For any \( Y \):

\[ \min_{a} \mathbb{E} \left[ (X - a)^2 \right] \bigg| Y = y \]

Solve simpler problem

\[ \min_{a} \mathbb{E} \left[ (X - a)^2 \right]. \]

Moments or inference around \( a \).
find a point \( a \) to minimize moment of inertia around \( a \).

center of mass:

\[
Y = E[X]
\]

\[
\min_a E \left[ (X - a)^2 \right] \Rightarrow a^* = E[X].
\]

\[
= \text{Var}(X) .
\]

back to:

\[
\min_a E \left[ (X - a)^2 \bigg| Y = y \right] = \text{Var}(X \bigg| Y = y)
\]

\[
a^* = E \left[ X \bigg| Y = y \right].
\]

\[
\Rightarrow \hat{X}_{\text{MMSE}}(y) = E \left[ X \bigg| \hat{Y} = y \right].
\]

short form:

\[
\hat{X}_{\text{MMSE}}(y) = E \left[ X \bigg| Y \right] .
\]
True for any channel and any prior.

Example 0: How to compute \( \mathbb{E}[X|Y] \) if \((X,Y)\) is jointly Gaussian?

\[ f(x,y) \]

\[ \frac{f(x,y)}{f(y)} = f(x|y) \]

Claim: Conditional on \( Y = y \), \( X \) is Gaussian.

Theorem: I can always write \( X \) as

\[ X = \alpha Y + \Delta \]

for some \( \alpha \), such that \( \Delta \) is Gaussian and independent of \( Y \).

From Theorem: 2) Claim follows

\[ \mathbb{E}[X|Y=y] = \alpha y + \mathbb{E}[\Delta] \]
Proof of theorem:

pick any \( \alpha \)

\[
\Delta = X - \alpha Y.
\]

\( \Delta \) is Gaussian

(\( \Delta, Y \)) is jointly Gaussian

\( \Delta, Y \) are independent if and only if \( \Delta, Y \) are uncorrelated.

Assume zero mean for \( \Delta, Y \).

\[
E[\Delta \cdot (Y)] = 0 \rightarrow \text{solve for } \alpha
\]

\[
E[(X - \alpha Y)Y] = 0
\]

\[
\Rightarrow E[X Y] - \alpha E[Y^2] = 0
\]

\[
\Rightarrow \alpha = \frac{E[X Y]}{E[Y^2]} \rightarrow \text{covariance } X, Y
\]

\[
\Rightarrow \alpha = \frac{\text{var } Y}{E[Y^2]}
\]
\[ \Delta = X - \alpha^* Y, \quad \alpha^* = \frac{E[xy]}{E[y^2]} \]

Then \( \Delta, Y \) are independent

\[ X = \alpha^* Y + \Delta \]  \[\sim\] Gaussian

\[ \text{Condition on } Y = y, \quad X \sim N(\alpha^* y, \text{Var}[Y] - \frac{\text{Cov}(x,y)^2}{\text{Var}[Y]^2}) \]

\[ \text{Var}[X] = (\alpha^*)^2 \text{Var}[Y] + \text{Var}[\Delta] \]

\[ \text{Var}[\Delta] = \text{Var}[X] - (\alpha^*)^2 \text{Var}[Y] \]

\[ \text{Var}[\Delta] = \text{Var}[X] - \frac{\text{Cov}(x,y)^2}{\text{Var}[Y]^2} \cdot \text{Var}[Y] \]

\[ \text{Var}[\Delta] = \text{Var}[X] - \frac{\text{Cov}(x,y)^2}{\text{Var}[Y]} \]
E[\sqrt{X}] = \alpha \ast Y = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \cdot Y

\text{linear function of } Y.