Recap:

\[ Y = h^T X + W \]

- \( X, W \) independent
- \( X \sim \mathcal{N}(0, \beta^2) \)
- \( W \sim \mathcal{N}(0, \sigma^2 I) \)

\[ V = h^T Y \text{ is a sufficient statistic} \]

\[ V = h^T (h^T X + W) = h^T h^T X + h^T W \]

\[ h^T W \sim \mathcal{N}(0, \beta^2 \sigma^2) \]

\[ \|h\| = 1 \]

\[ V^* = (h^*)^T Y \]
Why $V^\top$ is not useful for detecting $X$ given $Y$?

$V = \begin{bmatrix} h^\top Y = h^\top (h^\top X + \tilde{W}) = X + h^\top \tilde{W} = X + \tilde{W} \\
V = \begin{bmatrix} h \end{bmatrix}^\top Y = \begin{bmatrix} h \end{bmatrix}^\top (h^\top X + \tilde{W}) = 0 + \tilde{W} \tilde{W} = \begin{bmatrix} h \end{bmatrix}^\top \tilde{W} \tilde{W} = h^\top \tilde{W}$

($V$ is a suff. stat. if the likelihood ratio can be computed from $V$ without $Y$)

$W \sim N(0, \Sigma_{\tilde{W}})$

$\tilde{W} = \begin{bmatrix} h \end{bmatrix}^\top W$

$\tilde{W}_2 = h_{11}^\top W$

$W = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} h \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} W$
\( \tilde{W} \) is i.i.d., \( \tilde{W} \sim N(0, \sigma^2 I) \)
\( \tilde{W}, \tilde{W} \) are independent!

1) \( V^+ \) has no \( X \)
2) \( V^+ \) is independent of \( \tilde{W} \)
3) \( V^+ \) is useless.

Recap:
A suff. statistic \( V \)
1) can compute LR.
2) can do optimal detection.
3) given \( \tilde{W}, \tilde{W} \) and \( \tilde{W} \) are independent.

\( \begin{align*}
X & \xrightarrow{\text{Channel}} Y \rightarrow V \rightarrow X \\
\uparrow & \uparrow \uparrow \uparrow \\
V &= \mathcal{D}(Y)
\end{align*} \)

Given \( Y \), \( X \) and \( V \) are independent \( \checkmark \)

Given \( V \), \( X \) and \( Y \) are independent \( \checkmark \)
Given $V$, $X$ and $Y$ are independent, so I can throw $Y$ away!

$X$ and $Y$ are independent

$Y$ itself is a suff. statistic!

Given $Y$, $X$ and $Y$ are indp. $\checkmark$

Conditional independence:

$$f_{X,Y|V} = f_{X|V} \cdot f_{Y|V}$$

Interference example:

$$Y = hX + gZ + W, \quad W \sim N(0, \sigma_w^2)$$

signal

interference

dir.
dir.
\[ V = h^T Y \text{ is it a sufficient statistic?} \]

\[ V^+ = (h^+)^T Y = (h^+)^T (h X + g Z + W) \]

\[ = (h^+)^T g Z + (h^+)^T W = (h^+)^T \tilde{g} Z + \tilde{W}, \]

\[ V = X + h^T g Z + h^T W = X + \tilde{g} Z + \tilde{W}, \]

\( \tilde{W}_1 \) and \( \tilde{W}_2 \) are independent

In general, \( V^+ \) is useful, except for the special case \( (h^+)^T g = 0 \).

\[ V = h^T Y \rightarrow \text{projection of } V \text{ onto span}(g, h) \]

\[ \tilde{V} = g^T Y \]

Claim \((V, \tilde{V})\) is a sufficient statistic.

1. Signal in the plane
2. Interference \( Z \) in the plane
3. Noise \( W \) in plane is independent of noise in the plane.
component $\gamma$ plane has no infinite.

For detection,

$$W \rightarrow \hat{W}_1 = h^T W$$

$$\hat{W}_2 = (h g^T)^T W$$

$$\hat{W}_3 = a^T W$$

$$V = h^T Y = X + (h g^T Z + \hat{w}_1)$$

$$\tilde{V} = g^T Y = (g^T h X + \operatorname{diag} Z + \hat{w}_1')$$

$$= \begin{bmatrix} 1 \\ \hline g^T h \end{bmatrix} X + U = h^T X + \begin{bmatrix} 0 \\ \hline U \end{bmatrix}$$

$$U = \begin{bmatrix} h g^T Z + \hat{w}_1 \\ \hline \operatorname{diag} Z + \hat{w}_1' \end{bmatrix}$$

ex. $Y = h^T X + \tilde{W}$, $\tilde{W} \sim \mathcal{N}(0, \sigma^2 I)$