1. **Multiple Hypothesis Testing**  Consider a binary hypothesis testing problem, and denote the hypotheses as $X = 1$ and $X = -1$. Let $a = (a_1, a_2, \ldots, a_n)^T$ be an arbitrary real $n$-vector and let the observation be a sample value $y$ of the random vector $Y = Xa + Z$, where $Z \sim \mathcal{N}(0, \sigma^2[I_n])$ and $[I_n]$ is the $n \times n$ identity matrix. Assume that $Z$ and $X$ are independent.

(a) Recall the ML decision rule and find the probabilities of error $\Pr(e|X = 0)$ and $\Pr(e|X = 1)$ in terms of the function $Q(x)$.

Now suppose a third hypothesis, $X = 0$, is added to the situation of (a). Again the observation random vector is $Y = Xa + Z$, but here $X$ can take on values $-1, 0, \text{ or } +1$.

(b) How can we generalize the likelihood ratio to test three hypotheses, so that we can use this generalization to compute the ML decision?  

*(Hint: Can we use two ratios instead of one?)*

(c) Using part (b), find a low-dimensional sufficient statistic such that the ratios in part (b) can be calculated using the sufficient statistic. How many dimensions does your sufficient statistic have?

(d) Find the ML decision rule for this situation and find the probabilities of error, $\Pr(e|X = x)$ for $x = -1, 0, +1$.

2. **Poisson Hypothesis Testing**  
A sales executive hears that one of his salespeople is routing half of his incoming sales to a competitor. In particular, arriving sales are known to be Poisson at rate one per hour, i.e. the inter-arrival time is exponentially distributed with mean 1. According to the report (which we view as hypothesis $X = 1$), every second arrival is routed to the competitor; thus under hypothesis 1 the inter-arrival density for successful sales is $f(y|X = 1) = ye^{-y} (y \geq 0)$. The alternative hypothesis ($X = 0$) is that the rumor is false and the inter-arrival density for successful sales is $f(y|X = 0) = e^{-y} (y \geq 0)$. Assume that, a priori, the hypotheses are equally likely. The executive, a recent student of stochastic processes, explores various alternatives for choosing between the hypotheses; he can only observe the times of successful sales, however.

Starting with a successful sale at time 0, let $S_i$ be the arrival time of the $i$-th subsequent successful sale. The executive observes $S_1, S_2, \ldots, S_n$ $(n \geq 1)$ and chooses the maximum a posteriori probability hypothesis given this observation.

(a) Find the joint probability densities $(S_1, \ldots, S_n)$ given $X = 1$ and given $X = 0$ and give the MAP decision rule.

(b) What is a low-dimensional sufficient statistic in this case?