1. Deeper Kalman filter recursions  Reformulate the scalar system:

\[
X_0 \sim \mathcal{N}(0, 1)
\]

\[
X_1 = X_0 + W_1
\]

\[
X_n = X_{n-1} + 0.5X_{n-2} + W_n, \quad n = 2, 3, \ldots
\]

\[
Y_n = X_n + 0.4X_{n-1} + Z_n, \quad n = 1, 2, \ldots
\]

as a vector dynamical system. Here, \(W_n\)'s and \(Z_n\)'s are independent and \(\mathcal{N}(0, 1)\) distributed, and independent of \(X_0\) and \(X_1\).

**Solution:** By setting \(\tilde{X}_n = \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix}\) for \(n = 1, 2, \ldots\), one gets the system dynamics in the vector form as follows

\[
\tilde{X}_1 \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right),
\]

\[
\tilde{X}_n = \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ X_{n-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_n,
\]

\[
Y_n = \begin{bmatrix} 1 & 0.4 \end{bmatrix} \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} + \begin{bmatrix} 1 \\ D \end{bmatrix} Z_n.
\]

2. Signals and Systems  Consider the system with input \(x_n\) and output \(y_n\) given by

\[
y_n = \frac{1}{2} y_{n-1} + x_n.
\]

(a) The initial condition is: if \(x_n = 0 \forall n < N\), then \(y_n = 0 \forall n < N\). Is this an LTI system?

(b) The initial condition is \(y_n = 0\) for \(n < 0\). Is this an LTI system?

(c) What is the impulse response of the LTI version of this system?

(d) If we give the input

\[
x_n = u_n = \begin{cases} 
0 & n < 0 \\
1 & n \geq 0
\end{cases}
\]

calculate the output \(y_n\) of the LTI system.

(e) Verify that you get the same result as part (c) if you convolve the input and the impulse response.

**Solution:**
(a) Clearly, this is a linear system. If the input $x_n$ gives an output $y_n$, then for an input $\alpha x_n$, we also have $\alpha y_N = \frac{1}{2} \alpha y_{n-1} + \alpha x_n$, so the output must be $\alpha y_n$. Further, if input $x_n$ gives output $y_n$ and input $w_n$ gives output $z_n$, then the input $x_n + w_n$ must give the output $y_n + z_n$ because

$$y_n = \frac{1}{2} y_{n-1} + x_n, \quad z_n = \frac{1}{2} z_{n-1} + w_n \implies (y_n + z_n) = \frac{1}{2} (y_{n-1} + z_{n-1}) + (x_n + w_n)$$

by simply adding the two equations.

This is a time-invariant system. Suppose the input $x_n$ gives the output $y_n$. Then it is easy to see that the signal $x_{n-N}$ gives the output $y_{n-N}$. This is because the initial condition shifts along with the signal too, and since the recursion is valid for all $n$, we also have $y_{n-N} = \frac{1}{2} y_{n-N-1} + x_{n-N}$.

(b) This is linear for the same reasons as in the previous part, but not time-invariant. Suppose the output of $x_n$ is $y_n$ and $y_N \neq 0$ for some $N$. Then on input signal $x'_n = x_{n+N+1}$, the output must satisfy $y'_n = 0$ due to the initial condition. So, $y'_n \neq y_{n+N+1}$ at $n = -1$.

(c) We will calculate the impulse response for the system in (a). The input signal is

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

From the initial condition, $y_n = 0$ for $n < 0$. Then, we can work out the recursion equation as follows:

$$y_0 = \frac{1}{2} y_{-1} + x_0 = 1,$$
$$y_1 = \frac{1}{2} y_0 + x_1 = \frac{1}{2},$$
$$y_2 = \frac{1}{2} y_1 + x_2 = \frac{1}{4},$$
$$\vdots$$
$$y_n = \frac{1}{2} y_{n-1} + x_1 = \left(\frac{1}{2}\right)^n.$$

(d) Again, we can simply work out the recursion as follows:

$$y_0 = \frac{1}{2} y_{-1} + x_0 = 1,$$
$$y_1 = \frac{1}{2} y_0 + x_1 = \frac{1}{2} + 1,$$
$$y_2 = \frac{1}{2} y_1 + x_2 = \frac{1}{4} + \frac{1}{2} + 1,$$
$$\vdots$$
$$y_n = \frac{1}{2} y_{n-1} + x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} = 2 - \left(\frac{1}{2}\right)^n.$$

(e) Here, we show that $y_n = h_n \ast u_n$.

$$h_n \ast u_n = \sum_{k=-\infty}^{\infty} h_k u_{n-k} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = 2 - \left(\frac{1}{2}\right)^n.$$