1. An integrated Poisson process. Let $N(t)$ denote a Poisson process with rate $\lambda > 0$, and let $Y(t) = \int_0^t N(s)ds$.

   a. Sketch a typical sample path of $Y(t)$.

   b. Compute the mean function $\mu_Y(t)$, for $t \geq 0$.

   c. Compute $\text{Var}(Y(t))$ for $t \geq 0$.

Solution

a. An example with $\lambda = 1$ is shown in the figure below.

![Sample Path of Integrated Poisson Process](image)

b. We can compute the mean function as

$$
\mu_Y(t) = \mathbb{E} \left[ \int_0^t N(s) \, ds \right] = \int_0^t \mathbb{E}[N(s)] \, ds = \int_0^t \lambda s \, ds = \frac{\lambda}{2} t^2.
$$
c. In the class notes on the integrator system, we have seen that

\[ R_Y(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} R_N(\tau_1, \tau_2) \, d\tau_2 \, d\tau_1. \]

We can compute the variance of \( Y(t) \) as

\[ \text{Var}(Y(t)) = R_Y(t, t) - \mu_Y(t)^2 \]

\[ = \int_0^t \int_0^t (\lambda \min\{\tau_1, \tau_2\} + \lambda^2 \tau_1 \tau_2 \, d\tau_2 \, d\tau_1 - \mu_Y(t)^2 \]

\[ = \int_0^t \int_0^t \lambda \min\{\tau_1, \tau_2\} \, d\tau_2 \, d\tau_1 \]

\[ = \int_0^t \left( \int_0^{\tau_1} \lambda \min\{\tau_1, \tau_2\} \, d\tau_2 + \int_{\tau_1}^t \lambda \min\{\tau_1, \tau_2\} \, d\tau_2 \right) \, d\tau_1 \]

\[ = \int_0^t \left( \int_0^{\tau_1} \lambda \tau_2 \, d\tau_2 + \int_{\tau_1}^t \lambda \tau_1 \, d\tau_2 \right) \, d\tau_1 \]

\[ = \int_0^t \left( \frac{\lambda}{2} \tau_1^2 + \lambda \tau_1 t - \lambda \tau_1^2 \right) \, d\tau_1 \]

\[ = \int_0^t \left( \lambda \tau_1 t - \frac{\lambda}{2} \tau_1^2 \right) \, d\tau_1 \]

\[ = \frac{\lambda t^3}{2} - \frac{\lambda t^3}{6} \]

\[ = \frac{\lambda t^3}{3}. \]

**Remark:** Subtracting \( \mu_Y(t)^2 \) above cancels with the integral over \( \lambda^2 \tau_1 \tau_2 \). In fact,

\[ C_Y(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} C_N(\tau_1, \tau_2) \, d\tau_2 \, d\tau_1. \]

where \( C_Y(t_1, t_2) \) and \( C_N(t_1, t_2) \) are the autocovariance functions (as opposed to autocorrelation functions).

2. **Mixture process.** Let \( X(t) \) and \( Y(t) \) be two zero-mean WSS processes with autocorrelation functions \( R_X(\tau) \) and \( R_Y(\tau) \), respectively. Define the process

\[ Z(t) = \begin{cases} X(t) & \text{with probability } \frac{1}{2} \\ Y(t) & \text{with probability } \frac{1}{2} \end{cases} \]

a. Find the mean and autocorrelation functions for \( Z(t) \).

b. Is \( Z(t) \) a WSS process? Justify your answer.

**Solution**
a. The mean and autocorrelation functions of the mixture process $Z(t)$ are:

$$
\mu_Z(t) = E[Z(t)] = E(Z \mid Z = X)P\{Z = X\} + E(Z \mid Z = Y)P\{Z = Y\} = \frac{1}{2}(\mu_X + \mu_Y) = 0
$$

$$
R_Z(t + \tau, t) = E[Z(t + \tau)Z(t)] = \frac{1}{2} E[X(t + \tau)X(t)] + \frac{1}{2} E[Y(t + \tau)Y(t)] = \frac{1}{2}(R_X(\tau) + R_Y(\tau))
$$

b. Since $\mu_Z(t)$ is independent of time and $R_Z(t + \tau, t)$ depends only on $\tau$, we conclude that $Z(t)$ is WSS.

3. Mean ergodicity. Which of the following zero-mean WSS stationary processes are mean ergodic. Justify your answers.

a. $X(t)$ has autocorrelation function $R_X(\tau) = 2 - |\tau|$ for $|\tau| \leq 1$, $R_X(\tau) = 0$, otherwise.

b. $X(t)$ has autocorrelation function $R_X(\tau) = 2 - |\tau|$ for $|\tau| \leq 1$, $R_X(\tau) = 1$, otherwise.

c. $X(t)$ has autocorrelation function $R_X(\tau) = \cos(20\pi\tau)e^{-10|\tau|}$.

Solution

In (a), for $\tau \geq 1$ $R_X(\tau) = 0$, which implies that for $t \geq 1$:

$$
\int_0^t (t - \tau)R_X(\tau)d\tau = \int_0^1 (t - \tau)R_X(\tau)d\tau = \frac{t}{2} - \frac{1}{6}.
$$

As a result $X(t)$ is mean ergodic, since:

$$
\lim_{t \to \infty} \frac{2}{t^2} \int_0^t (t - \tau)R_X(\tau)d\tau = 0
$$

and $X(t)$ has mean zero.

In (b), for $\tau \geq 1$ $R_X(\tau) = 0$, which implies that for $t \geq 1$:

$$
\int_0^t (t - \tau)R_X(\tau)d\tau = \int_0^1 (t - \tau)R_X(\tau) + \int_1^t (t - \tau)d\tau = \frac{3t}{2} - \frac{2}{3} + \frac{t^2}{2} - t + \frac{1}{2}.
$$

As a result $X(t)$ is not mean ergodic, since:

$$
\lim_{t \to \infty} \frac{2}{t^2} \int_0^t (t - \tau)R_X(\tau)d\tau = 1
$$

and $X(t)$ has mean zero.

In (c), let us upper bound the absolute value of the following integral:

$$
\left|\int_0^t (t - \tau)R_X(\tau)d\tau\right| \leq \int_0^t |(t - \tau)\cos(2\pi\tau)|e^{-10\tau}d\tau
$$

$$
\leq \sup_{0 \leq \tau \leq t} |(t - \tau)\cos(2\pi\tau)| \int_0^t e^{-10\tau}d\tau
$$

$$
\leq t \frac{1 - e^{-10t}}{10}.
$$
This implies that $X(t)$ is mean ergodic, since:

$$\lim_{t \to \infty} \frac{2}{t^2} \int_0^t (t - \tau) R_X(\tau) d\tau = 0$$

and $X(t)$ has mean zero.