1. **Two-dimensional random walk on the integer grid.** Consider the random walk vector process 
\[ [X_n, Y_n], n \geq 0, \] on the two dimensional integer grid \( \{\ldots, -2, -1, 0, +1, +2, \ldots \}^2 \) defined as follows: Let \( U_n, n \geq 1, \) be an i.i.d. process such that 
\[ p_{U_1}(+1) = p_{U_1}(-1) = \frac{1}{2} \] and \( V_n, n \geq 1, \) be a Bernoulli process with parameter \( \frac{1}{2}, \) where the \( U_n \) and \( V_n \) processes are independent. At time \( n = 0, [X_0, Y_0] = [0, 0], \) and at time \( n \geq 1, \)
\[ [X_n, Y_n] = [X_{n-1}, Y_{n-1}] + [V_n \cdot U_n, (1 - V_n) \cdot U_n]. \]

a. Sketch a sample path from this vector process.

b. What is the probability of the event \( \{[X_{2n}, Y_{2n}] = [n, n]\}. \)

c. Are the processes \( X_n \) and \( Y_n, n \geq 1, \) independent increment?

d. Are the processes \( X_n \) and \( Y_n, n \geq 1, \) independent? Justify your answer.

e. Are the processes \( X_n \) and \( Y_n, n \geq 1, \) uncorrelated? You need to evaluate the cross-correlation function \( R_{XY}(n_1, n_2). \)

f. Does the sequence of random vectors \( \frac{1}{\sqrt{n}} \cdot [2X_n, 4Y_n] \) converge as \( n \to \infty? \) Justify your answer and if the sequence converges, specify the sense of the convergence and the limit.

g. Define the random process \( W_n = e^{X_n + Y_n}, n \geq 1. \) Is \( W_n \) an independent increment process? Is it a Markov process? Justify your answers.

**Solution**

a. Note that at each time \( n, \) a step \( U_n \) is taken either in the \( x \) direction (if \( V_n = 1 \)) or \( y \) direction (if \( V_n = 0 \)). A sample path is plotted in Figure 1.

b. The probability of the event is: \( \left( \frac{2^n}{n} \right) \cdot 2^{-4n}. \)

c. From the definition, the processes \( X_n \) and \( Y_n \) are independent increment.

d. The processes \( X_n \) and \( Y_n \) are not independent. For example \( X_1 = 1, \) we know that \( P\{Y_1 = 0 | X_1 = 1\} = 1, \) but \( P\{Y_1 = 0\} = \frac{1}{2}. \)

e. To show that the processes are uncorrelated, we need to show that \( R_{XY}(n_1, n_2) = 0 \) for all
Figure 1: Sample functions of $X(t)$.

\[ n_1, n_2 \geq 1. \text{ Assume } n_1 \leq n_2, \text{ then,} \]

\[ R_{XY}(n_1, n_2) = E(X_{n_1}Y_{n_2}) \]
\[ = E(X_{n_1}(Y_{n_2} - Y_{n_1} + Y_{n_1})) \]
\[ = E(X_{n_1}Y_{n_1}) \quad (X_{n_1} \text{ and } Y_{n_2} - Y_{n_1} \text{ are independent and zero mean}) \]
\[ = E\left( \left( \sum_{i=1}^{n_1} V_i U_i \right) \left( \sum_{i=1}^{n_1} (1 - V_i) U_i \right) \right) \]
\[ = E\left( \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} V_i (1 - V_j) U_i U_j \right) \]
\[ = \sum_{i=1}^{n_1} E(V_i (1 - V_i) U_i^2) \]
\[ = n_1 \cdot (E(V_1 U_1^2) - E(V_1^2 U_1^2)) \]
\[ = n_1 \cdot (E(V_1) E(U_1^2) - E(V_1^2) E(U_1^2)) = 0. \]

f. We can write $X_n = \sum_{i=1}^{n_1} V_i U_i$ and $Y_n = \sum_{i=1}^{n_1} (1 - V_i) U_i$, thus

\[ \frac{1}{\sqrt{n}} \cdot [2X_n \quad 4Y_n] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n_1} [2V_i U_i \quad 4(1 - V_i) U_i]. \]

Since the vectors inside the sum are i.i.d., by the CLT for random vectors, their normalized sum converges in distribution to a Gaussian RV. Clearly, the mean of the Gaussian is $[0 \ 0]$. To find its covariance matrix, note that it is as same as the covariance of $[2V_i U_i \ 4(1 - V_i) U_i]$. But we know that $V_i U_i$ and $(1 - V_i) U_i$ are uncorrelated, thus the covariance is a diagonal
matrix with diagonal elements

$$E[(2V_i U_i)^2] = 4 \cdot \frac{1}{2} \cdot 1 = 2$$

$$E[(4(1 - V_i) U_i)^2] = 16 \cdot \frac{1}{2} \cdot 1 = 8.$$ 

In summary

$$\frac{1}{\sqrt{n}} \cdot [2X_n \quad 4Y_n] \to \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \right) \text{ in distribution.}$$

g. The random process $W_n$ is not an independent increment process. To show this, consider the increments $W_1 = e^{U_1}, W_2 - W_1 = e^{U_1}(e^{U_2} - 1)$. Now, conditioned on $U_1 = 1, W_2 - W_1 = e(e - 1)$ with probability $\frac{1}{2}$ and $(1 - e)$ with probability $\frac{1}{2}$. On the other hand, unconditionally, $W_2 - W_1 = e(e - 1)$ with probability $\frac{1}{4}$, $(1 - e^{-1})$ with probability $\frac{1}{4}$, $e^{-1}(e^{-1} - 1)$ with probability $\frac{1}{4}$. The process $W_n$ is Markov because $W_n = W_{n-1} \cdot e^{U_n}$, so conditioned on $W_{n-1}, W_n$ and $W_1, W_2, \ldots, W_{n-2}$ are independent.

2. Conditional distributions.

a. Let $X, Z \sim \mathcal{N}(0, 1)$ and independent. Let $Y = 2X + Z$. Compute the conditional distribution of $X$ given $Y = y$.

b. Let $X, Z_1, Z_2, \ldots, Z_n \sim \mathcal{N}(0, 1)$ and are mutually independent. Let $Y_i = 2X + Z_i$, Compute the conditional distribution of $X$ given $Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n$. What happens to the conditional distribution as $n \to \infty$? Give an intuitive explanation.

Solution

a. We know that

$$\text{Var}[X] = 1, \quad \text{Var}[Y] = 5, \quad \text{Cov}(X, Y) = 2.$$ 

Then

$$E[X|Y = y] = E[X] + \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} [y - E[Y]] = \frac{2}{5} y$$

$$\text{Var}[X|Y = y] = \text{Var}[X] - \frac{\text{Cov}(X, Y)^2}{\text{Var}[Y]} = \frac{1}{5}.$$ 

Then we know $X|Y = y \sim \mathcal{N}(\frac{2}{5} y, \frac{1}{5})$.

b. Let $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$. Then we know $\bar{Y}$ is a sufficient statistic of $Y_1, \ldots, Y_n$ with respect to $X$. Then we have the following Markov chain:

$$X - \bar{Y} - Y_1, \ldots, Y_n,$$

giving that

$$[X|Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n] \overset{\text{dist}}{=} [X|\bar{Y} = \bar{y}].$$

Note that

$$\text{Var}[\bar{Y}] = 4 + \frac{1}{n}, \quad \text{Cov}(X, \bar{Y}) = 2.$$ 

We finally have

$$[X|Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n] \overset{\text{dist}}{=} [X|\bar{Y} = \bar{y}] = \mathcal{N}(\frac{2n}{4n+1} \bar{y}, \frac{1}{4n+1}).$$
where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$. 