1. Distribution tilting

(a) If $X$ is a continuous random variable having cumulative distribution function (CDF) $F$, show that the random variable $Y$ defined to be $F(X)$ is uniformly distributed in $(0, 1)$.

(b) Use this to come up with a method to sample a random variable $X$ with CDF $F$ if you can only sample a random variable $Y$ which is uniformly distributed in $(0, 1)$.

2. Law of Large Numbers for the Median

In class, we have seen that the LLN shows convergence of the sample average to the mean. Here we will see that the LLN allows us to estimate the entire CDF $F_X(x)$ of a random variable $X$, and the median of $X$ using several independent samples.

(a) Let $X_1, \ldots, X_n$ be i.i.d. random variables with the CDF $F_X(x)$. For any given $y$, let $I_j(y)$ be the indicator function for the event $\{X_j \leq y\}$, i.e.

$$I_j(y) = \begin{cases} 1 & X_j \leq y \\ 0 & \text{otherwise} \end{cases}$$

Note that $I_1(y), \ldots, I_n(y)$ are also i.i.d. random variables. State the law of large numbers for the random variables $I_j(y)$.

(b) Suggest a method to estimate the median of $X$ using the samples $X_1, \ldots, X_n$. Assume that $X$ is a continuous random variable and that its PDF is positive in an open interval around the median.

3. Fun with Gaussian random variables.

Let $X_1, X_2, X_3 \sim \mathcal{N}(0, 1)$. Compute the following quantities:

(a) The distribution of $Y_1 = 2X_1 + 3$.

(b) The distribution of $Y_2 = X_1 + 2X_2$ and the distribution of $Y_3 = X_1 - X_3$.

(c) The correlation between $Y_2$ and $Y_3$:

$$\text{corr}(Y_2, Y_3) = \frac{\text{Cov}(Y_2, Y_3)}{\sqrt{\text{Var}(Y_2)\text{Var}(Y_3)}}$$

4. Eigenvalues and eigenvectors of a symmetric matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$ 

(a) Without computing the eigenvalues, what is the sum and product of the eigenvalues of $A$?

(b) Now compute the eigenvalues of $A$. 

References