1. Kalman filter for location tracking

Consider a truck on frictionless, straight rails. Initially, the truck is stationary at location $L_0 = 0$ and moving with velocity $V_0 = 1$, but it is buffeted by random uncontrolled forces. We measure the position of the truck every $\Delta t = 1$ seconds, but these measurements are imprecise; we want to maintain a model of the truck’s location $L_t$ and its velocity $V_t$.

Specifically, assuming at time $t = 0$, the initial state of the truck is $L_0 = 0, V_0 = 1$. Between $t-1$ and $t$, the velocity is subject to a constant acceleration $A_{t-1} \sim \mathcal{N}(0, \sigma_a^2)$. Also at time $t$, we take a noisy observation $Y_t = L_t + Z_t$, where $Z_t \sim \mathcal{N}(0, \sigma_z^2)$.

Formulate this problem as a vector Kalman filter estimation problem, with $X_t = [L_t, V_t]^T$.

**Solution:**

According to Newton’s law, the dynamics can be written as

$$L_t = L_{t-1} + V_{t-1} + \frac{1}{2} A_{t-1}$$

$$V_t = V_{t-1} + A_{t-1}$$

The observation can be written as

$$Y_t = L_t + Z_t$$

Define the state at time $t$, $X_t = [L_t, V_t]^T$. Then in vector form, the above relations are

$$X_t = \begin{bmatrix} L_t \\ V_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X_{t-1} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} A_{t-1}$$

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} X_t + Z_t.$$ 

Through this formulation, even though we only observe a noisy version of $L_t$, we can use this to estimate both $L_t$ and $V_t$ since they depend on each other through the $A_{t-1}$’s.

2. Kalman filter with fading coefficient

Consider the scalar Kalman filter specified by $X_1 \sim \mathcal{N}(\bar{X}, \sigma_X^2)$, and

$$X_n = \alpha X_{n-1} + W_n \text{ for } n = 2, 3, ...$$

$$Y_n = h X_n + Z_n \text{ for } n = 1, 2, 3, ...$$

where $W_n \sim \mathcal{N}(0, \sigma_w^2)$ and $Z_n \sim \mathcal{N}(0, \sigma_z^2)$ are IID and independent each other and $X_1$. How do the following things change from the ones derived in class, in the presence of $h$?

(a) The estimate $\hat{X}_1(y_1)$ and its mean squared error, i.e. $v_1^2$

(b) The estimate $\hat{X}_2(y_1, y_2)$ and its mean squared error, i.e. $v_2^2$

**Solution:**

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1Example taken from https://en.wikipedia.org/wiki/Kalman_filter#Example_application
(a) For \( n = 1 \), we have \( X_1 \sim \mathcal{N}(\bar{X}, \sigma^2_{X_1}) \) and \( Y_1 = hX_1 + Z_1 \). Thus,

\[
\hat{Y}_1 = h\bar{X}, \quad \text{Var}(Y_1) = h^2\sigma^2_{X_1} + \sigma_z^2.
\]

\[
\text{Cov}(X_1, Y_1) = \mathbb{E}[(X_1 - \bar{X})(Y_1 - \bar{Y}_1)] = h\mathbb{E}[(X_1 - \bar{X})(X_1 + Z_1 - h\bar{X})] = h\mathbb{E}[(X_1 - \bar{X})(X_1 - \bar{X})] + \mathbb{E}[(X_1 - \bar{X})Z_1] = h\sigma^2_{X_1} + 0
\]

The estimator of \( X_1 \) given \( Y_1 \) is then

\[
\hat{X}_1(Y_1) = \mathbb{E}[X_1 \mid Y_1 = y_1] = \bar{X} + \frac{\text{Cov}(X_1, Y_1)}{\text{Var}(Y_1)}(Y_1 - \bar{Y}_1)
\]

\[
= \bar{X} + \frac{h\sigma^2_{X_1}}{h^2\sigma^2_{X_1} + \sigma_z^2}(Y_1 - h\bar{X})
\]

The mean squared error is

\[
v_1^2 = \text{Var}(X_1 \mid Y_1) = \text{Var}(X_1) - \frac{\text{Cov}(X_1, Y_1)^2}{\text{Var}(Y_1)}
\]

\[
= \sigma^2_{X_1} - \frac{h^2\sigma^4_{X_1}}{h^2\sigma^2_{X_1} + \sigma_z^2}
\]

\[
= \frac{\sigma^2_{X_1}\sigma_z^2}{h^2\sigma^2_{X_1} + \sigma_z^2}
\]

We can write this as

\[
\frac{1}{v_1^2} = \frac{1}{\sigma^2_{X_1}} + \frac{h^2}{\sigma_z^2}
\]

Note the differences between (1),(2),(3) and the corresponding equations derived in Lecture 13, in particular where the factor \( h \) appears.

From (3), you can see that \( v_1^2 \) is the same as in the case when the input is not scaled and the noise variance is \( \sigma_z^2/h^2 \) instead. This is a useful way to remember (3).

(b) With \( \hat{X}_1(y_1) \) and \( v_1^2 \) as in equations (1) and (2), we have

\[
\mathbb{E}[X_2 \mid Y_1 = y_1] = \alpha \hat{X}_1(y_1), \quad \text{Var}(X_2 \mid Y_1) = \alpha^2 v_1^2 + \sigma^2_w = u_1^2
\]

Thus, by using \( \mathbb{E}[X_2 \mid Y_1 = y_1] \) instead of \( \bar{X} \) and \( \text{Var}(X_2 \mid Y_1) \) instead of \( \sigma^2_{X_1} \), the estimate at \( n = 2 \) is

\[
\hat{X}_2(Y_1, Y_2) = \alpha \hat{X}_1(Y_1) + \frac{hu_1}{h^2 u_1^2 + \sigma_z^2}(Y_2 - h\alpha \hat{X}_1(Y_1))
\]

The mean squared error is

\[
v_2^2 = v_1^2 - \frac{h^2 u_1^4}{h^2 u_1^2 + \sigma_z^2}
\]

\[
= \frac{u_1^2 \sigma_z^2}{h^2 u_1^2 + \sigma_z^2}
\]
Again, this can be written as

$$\frac{1}{v_2^2} = \frac{1}{u_1^2} + \frac{h^2}{\sigma_z^2}$$

$$= \frac{1}{\alpha^2 v_1^2 + \sigma_w^2} + \frac{h^2}{\sigma_z^2}$$

(6)

We can generalize this to \( n > 2 \) as

$$u_{n-1}^2 = \alpha^2 v_{n-1}^2 + \sigma_w^2$$

(7)

$$\tilde{X}_n(Y_1, \ldots, Y_n) = \alpha \tilde{X}_{n-1} + \frac{hu_{n-1}^2}{h^2 u_{n-1}^2 + \sigma_z^2} (Y_n - h\alpha \tilde{X}_{n-1})$$

(8)

$$\frac{1}{v_n^2} = \frac{1}{\alpha^2 v_{n-1}^2 + \sigma_w^2} + \frac{h^2}{\sigma_z^2}$$

(9)

Again, compare these equations with the corresponding ones derived in Lecture 13 to see how they depend on \( h \).