Section 9
EE278: Introduction to Statistical Signal Processing (Fall 2020)
Monday, Nov 16, 2020 - 3 to 4 pm

1. Periodic Signal with Noise
Consider the process \( X_n = \cos \left( \frac{2\pi}{k} n + \Theta \right) \) where \( k \) is a fixed positive integer, and \( \Theta \) is a \( \text{Unif}[0, 2\pi) \) random variable. Let \( Z_n \) be a white Gaussian noise process, i.e. \( Z_n \) are i.i.d. \( \mathcal{N}(0, \sigma^2) \), independent of \( X_n \).

(a) Sketch a sample path of \( X_n \) for \( k = 2 \) and \( k = 5 \).

(b) Calculate the covariance function \( K_X(m) \) and the power spectral density \( S_X(f) \). Sketch them.

(c) Now consider the process \( Y_n = X_n + Z_n \). Sketch a sample path of \( Y_n \) for \( k = 2 \) and \( k = 5 \).

(d) Calculate the covariance function \( K_Y(m) \) and the power spectral density \( S_Y(f) \). Sketch them.

(e) Suppose you observe the signal \( Y_n \) and want to estimate \( k \). How can you estimate \( k \) directly from \( Y_n \)? How can you estimate \( k \) from \( K_Y(m) \) or \( S_Y(f) \)? Out of \( Y_n \), \( K_Y(m) \) and \( S_Y(f) \), from which of these can you get the best estimate for \( k \)?

(f) Suppose \( Z_n \) is not white but has covariance function \( K_Z(m) = \sigma^2 e^{-|m|} \). Repeat parts (d) and (e) for this case.

Solution:

(a) For any fixed \( k \), you can sketch a sample path by first choosing a random \( \Theta \in [0, 2\pi) \), sketching the graph of \( \cos \left( \frac{2\pi}{k} x + \Theta \right) \) vs \( x \), then sampling this graph at integer-valued points. For \( k = 2 \), this is just \( \cos(\pi n + \Theta) \) which we sketched in Section 8. See the sketches below.

(b) As seen in Section 8, \( X_n \) is a stationary process. Hence we can write its covariance function as \( K_X(m) \).

\[
K_X(m) = \mathbb{E}[X_0 X_m]
= \mathbb{E} \left[ \cos(\Theta) \cos \left( \frac{2\pi}{k} m + \Theta \right) \right]
= \frac{1}{2} \mathbb{E} \left[ \cos \left( \frac{2\pi}{k} m \right) + \cos \left( \frac{2\pi}{k} m + 2\Theta \right) \right] \quad \text{using} \ \cos(A) \cos(B) = \frac{1}{2} (\cos(A - B) + \cos(A + B))
= \frac{1}{2} \cos \left( \frac{2\pi}{k} m \right) + 0 \quad \text{since} \ \frac{2\pi}{k} m + 2\Theta \ \text{is uniform over a range of} \ 4\pi
\]

Take the Fourier transform of \( K_X(m) \) to calculate \( S_X(f) \). You can either recall the Fourier transform
of a cosine or follow the steps below.

\[
S_X(f) = \sum_{m=-\infty}^{\infty} K_X(m) e^{-j2\pi fm}
\]

\[
= \sum_{m=-\infty}^{\infty} \frac{1}{2} \cos \left(\frac{2\pi}{k} m\right) e^{-j2\pi fm}
\]

\[
= \frac{1}{4} \sum_{m=-\infty}^{\infty} \left( e^{-j\frac{2\pi}{k} m} + e^{+j\frac{2\pi}{k} m} \right) e^{-j2\pi fm} \quad \text{using } \cos(\theta) = \frac{1}{2} \left(e^{-j\theta} + e^{+j\theta}\right)
\]

\[
= \frac{1}{4} \left( \delta \left( f - \frac{1}{k} \right) + \delta \left( f + \frac{1}{k} \right) \right)
\]

Here, we evaluate \( S_X(f) \) only for \(-\frac{1}{2} \leq f \leq \frac{1}{2}\), since \( S_X(f) \) is periodic with period 1. See the sketches below.

(c) This is just a noisy observation of (a). See the sketches below.

(d) Recall that for the i.i.d. noise process \( Z_n \),

\[
K_Z(m) = \begin{cases} 
\sigma^2 & m = 0 \\
0 & m \neq 0 
\end{cases}
\]

\[
S_Z(f) = \sigma^2
\]
Further, since $X_n$ and $Z_n$ are independent,

\[ K_Y(m) = \begin{cases} \frac{1}{2} \cos \left( \frac{2\pi}{k} m \right) + \sigma^2 & m = 0 \\ 0 & m \neq 0 \end{cases} \]

\[ S_Y(f) = \frac{1}{4} \left( \delta(f - \frac{1}{k}) + \delta(f + \frac{1}{k}) \right) + \sigma^2 \]

(e) To find out the period $k$ from $Y_n$, we could count the time difference between two zero crossings. However this could be a noisy estimate as the noise can shift the zero crossings. We know that $K_Y(m)$ has a peak at $m = k$. Hence we can estimate $k$ as the index of the first peak in $K_Y(m)$ after $m = 0$. We also know that $S_Y(f)$ is a constant, except for two values of $f$ at which it is very large ($f = 1/k$ and $-1/k$). Thus we can estimate $1/k$ as the positive value of $f$ where $S_Y(f)$ is the largest. Thus $K_Y$ and $S_Y$ give us an estimate for $k$ which is robust no matter what the noise variance is!

(f) In this case, we have

\[ K_Y(m) = \frac{1}{2} \cos \left( \frac{2\pi}{k} m \right) + \sigma^2 e^{-|m|} \]

Let us first calculate the Fourier transform of $K_Z$.

\[ S_Z(f) = \sum_{m=-\infty}^{\infty} \sigma^2 e^{-|m|} e^{-j2\pi fm} \]

\[ = 2\sigma^2 \text{Re} \left\{ \sum_{m=0}^{\infty} e^{-(1+j2\pi f)m} \right\} - \sigma^2 \]

\[ = 2\sigma^2 \text{Re} \left\{ \frac{1}{1 - e^{-(1+j2\pi f)}} \right\} - \sigma^2 \]

\[ = \sigma^2 \left( \frac{1}{1 - e^{-1} e^{j2\pi f}} + \frac{1}{1 - e^{-1} e^{-j2\pi f}} \right) - \sigma^2 \]

\[ = \frac{2\sigma^2(1 - e^{-1} \cos(2\pi f))}{1 + e^{-2} - 2e^{-1} \cos(2\pi f)} - \sigma^2 \]

\[ = \frac{\sigma^2(1 - e^{-2})}{1 + e^{-2} - 2e^{-1} \cos(2\pi f)} \]
In this case too, it is unreliable to estimate $k$ from $Y_n$ directly. Even detecting the first peak of $K_Y(m)$ can be unreliable since the PSD of the noise can shift the peak of $K_Y(m)$. But $S_Y(f)$ is very large at two values of $f$ ($f = 1/k$ and $-1/k$) even in this case. Thus we can estimate $1/k$ as the positive value of $f$ where $S_Y(f)$ is the largest. This will still be a robust estimate no matter what the noise variance or PSD is.

In audio (speech or music) processing, the standard method to estimate the frequency of a periodic signal is to find the peaks of the power spectral density.