EE359 Discussion Session 7
Adaptive Modulation, MIMO systems

November 11, 2015
Outline

1. Recap

2. Adaptive transmit schemes
   - Variable rate variable power
   - Discrete rate variable power
   - Validity
   - Channel estimation errors

3. MIMO systems
   - Representation
   - Capacity
Last discussion session

- Midterm review

This session
- Adaptive modulation and adaptive power MQAM
- MIMO (multiple input multiple output) systems
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2 Adaptive transmit schemes
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3 MIMO systems
   - Representation
   - Capacity
Adapt to what?

Idea

Use the estimate of the instantaneous channel state fed back to the transmitter

Adapt what?

- Power
- Modulation (rate)
- Coding
- QoS (quality of service or BER $P_b$)
- ...

Problems

Estimate of the channel may be:

- Erroneous (due to receiver measurement error)
- Outdated (due to non zero delay of the feedback link)
Adapt what?

- Power
- Modulation
- Coding
- QoS (quality of service or BER $P_b$)
- ...
Adaptive schemes used in many high performance systems

Lots of tunable parameters

- Adaptive modulation: 802.11ac (latest commercial Wifi standard) uses BPSK, QPSK, 16 QAM, 64 QAM, 256 QAM
- Adaptive power: Used more in multiuser systems (e.g. power control in CDMA systems) (we are discussing point to point channels here)
- Adaptive rate coding: Also used for QoS requirements, e.g. 802.11ac has coding rates of $1/2, 2/3, 3/4, 5/6$
- Adaptive spatial streams: Different configurations for up to $8 \times 8$ MIMO
- Adaptive bandwidth: 802.11 has 20, 40, 60, 80 or 160 MHz channels
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Spectral efficiency and target BER

**Spectral efficiency**

Simply defined as the number of bits sent on average per unit bandwidth (i.e. $E_i[\log_2(M_i)]$ bits where an $M_i$-ary constellation is sent at time $i$)

**Target BER**

- Probability of error $P_b$ cannot be zero for practical systems
- For 1 symbol, SNR $\gamma$, *no coding*, nearest neighbour decoding for an MQAM constellation,

$$P_b \leq 0.2e^{-1.5\gamma M^{-1}}$$
Spectral efficiency and target BER

Spectral efficiency

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Target BER

- Probability of error $P_b$ cannot be zero for practical systems.
- For 1 symbol, SNR $\gamma$, no coding, nearest neighbour decoding for an MQAM constellation,

$$P_b \approx 0.2e^{-1.5\gamma \over M-1}$$
An equivalent way of looking at $P_b \approx 0.2e^{\frac{-1.5\gamma}{M-1}}$

- Given SNR $\gamma$, the maximum constellation size “supported” for BER $P_b$ is

$$M \leq 1 - \frac{1.5}{\ln(5P_b)\gamma}$$
An equivalent way of looking at \( P_b \approx 0.2e^{-1.5\gamma M^{-1}} \)

- Given SNR \( \gamma \), the maximum constellation size “supported” for BER \( P_b \) is
  \[
  M \leq 1 + K\gamma
  \]

Questions

- What happens when \( K > 1 \)?
- Is a bigger \( K \) better from a performance standpoint?
- What does \( K \) depend on/how do we improve \( K \)?
- Does smaller BER mean smaller or larger \( K \)?
An equivalent way of looking at $P_b \approx 0.2e^{-1.5\gamma M - 1}$

- Given SNR $\gamma$, the maximum constellation size “supported” for BER $P_b$ is
  $$M \leq 1 + K\gamma$$

**Questions**

- What happens when $K > 1$?
  Spectral efficiency greater than capacity

- Is a bigger $K$ better from a performance standpoint?
  Bigger $K$ means more spectral efficiency for a given power

- What does $K$ depend on/how do we improve $K$?
  Coding, BER, blocklength/latency/delay

- Does smaller BER mean smaller or larger $K$?
  Smaller $K$
Variable rate variable power MQAM

Idea
Maximize spectral efficiency (S.E.) subject to average power constraints

Math

\[
\text{maximize } P(\gamma) \mathbb{E}[\log_2(M)] \\
\text{s.t. } \mathbb{E}[P(\gamma)] = \bar{P}
\]

Solution
Optimal \( P(\gamma) \) given by

\[
P(\gamma)/\bar{P} = \begin{cases} 
1/\gamma_0 - 1/(\gamma K) & \gamma \geq \gamma_0/K \\
0 & \gamma < \gamma_0/K 
\end{cases}
\]

where \( \gamma_0 \) satisfies \( \mathbb{E}[(K/\gamma_0 - 1/\gamma)1(\gamma > \gamma_0/K)] = K \). Optimal S.E. is

\[
\mathbb{E}[\log_2(K\gamma/\gamma_0)1(\gamma > \gamma_0/K)]
\]
Variable rate variable power MQAM

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Maximize spectral efficiency (S.E.) subject to average power constraints

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\[
\text{maximize}_P P(\gamma) \ E[\log_2 (1 + KP(\gamma)/\bar{P})] \\
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More details about variable rate variable power MQAM

Comparison of solution with Shannon capacity expression

- Same waterfilling ideas work in this case
- “Rate” $M$ is now the size of the constellation
- Capacity expression if $K = 1$; does it imply BER for capacity is non zero?
- $K$ represents “power loss” due to BER requirement

Extension of BER constrained schemes

- Channel inversion: S.E. is $\log_2(1 + K\sigma)$, where $\sigma = 1/E[1/\gamma]$
- Truncated channel inversion: S.E. is $(1 - P(\gamma < \gamma_0)) \log_2(1 + K\sigma)$, where $\sigma = 1/E[1/\gamma 1(\gamma > \gamma_0)]$. 
More details about variable rate variable power MQAM

Comparison of solution with Shannon capacity expression

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Extension of BER constrained schemes

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<td>Rate cannot be continuous in practice</td>
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Continuous rate continuous power MQAM

Problem

$$\max_{P(\gamma)} \mathbb{E}[\log_2(M(\gamma))]$$

s.t.  $$\mathbb{E}[P(\gamma)] = \bar{P}, \quad P_b(\gamma) \leq P_b$$

The solution

Use waterfilling

Figure: $M$ versus $\gamma$
Discrete rate variable power MQAM

Problem

$$\max_{P(\gamma), M(\gamma)} E[\log_2(M(\gamma))]$$

s.t. $E[P(\gamma)] = \bar{P}$, $M(\gamma) \in \mathcal{M}$, $P_b(\gamma) \leq P_b$

Ideas to approximate solution

- Given available constellations $\mathcal{M} = \{M_1, \ldots, M_n\}$, choose a constellation which meets the $P_b$ target for a given channel state $\gamma$
- Reduce transmit power to keep BER constant if channel is good

Figure: $M$ versus $\gamma$
Discrete rate variable power MQAM

Problem

$$\max_{P(\gamma), M(\gamma)} E[\log_2 (M(\gamma))]$$
\[\text{s.t. } E[P(\gamma)] = \bar{P}, M(\gamma) \in \mathcal{M}, P_b(\gamma) \leq P_b\]

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Figure: \( M \) versus \( \gamma \)
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- Reduce transmit power to keep BER constant if channel is good

Figure: \( M \) versus \( \gamma \)
Power adaptation

Idea

Assuming that $P_b \approx 0.2e^{-\frac{1.5\gamma}{M-1}}$, for constant $M$, $P_b \downarrow$ as $\gamma \uparrow$

To conserve power simply reduce power if channel state is good!

\[ \frac{(M_3 - 1)/K}{\gamma_3} \quad \frac{(M_2 - 1)/K}{\gamma_2} \quad \frac{(M_1 - 1)/K}{\gamma_1} \]

Figure: $\gamma P(\gamma) / \bar{P}$ versus $\gamma$

Resultant spectral efficiency (S.E.)

\[ R/B = \sum_i \log_2(M_i)p(\gamma_i \leq \gamma < \gamma_{i+1}) \]
Power adaptation

Idea

Assuming that $P_b \approx 0.2e^{-1.5\gamma} \frac{1}{M-1}$, for constant $M$, $P_b \downarrow$ as $\gamma \uparrow$

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Resultant spectral efficiency (S.E.)

$$R/B = \sum_{i} \log_2(M_i)p(\gamma_i \leq \gamma < \gamma_{i+1})$$
Power adaptation

**Idea**

Assuming that $P_b \approx 0.2 e^{-1.5 \frac{\gamma}{M-1}}$, for constant $M$, $P_b \downarrow$ as $\gamma \uparrow$

To conserve power simply reduce power if channel state is good!

![Graph](image)

**Figure:** $P(\gamma)/\bar{P}$ versus $\gamma$

**Resultant spectral efficiency (S.E.)**

$$R/B = \sum_i \log_2(M_i)p(\gamma_i \leq \gamma < \gamma_{i+1})$$
And so on . . .

- What we have presented so far is achievable, but not necessarily optimal.
- Reduction in optimization complexity achieved by using insights from waterfilling solutions.
- Can also use insights from channel inversion/truncated channel inversion.
- Can extend ideas to discrete power, discrete rate.
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Using adaptive schemes

Idea

The channel cannot change too fast!

- Valid when time that channel “stays” in a particular state is much higher than several symbol times
- Calculation in terms of level crossing rates with Markov assumption (with jump only between adjacent regions)

Figure: Markov assumption \((R_0 = \{\gamma : \gamma < \gamma_0\}, R_1 = \{\gamma : \gamma_0 \leq \gamma < \gamma_1\}, \ldots)\)
Homework 6

Problem 1
1 Use the adaptive rate and power adaptation assuming continuous $M(\gamma)$ for first two parts
2 Use the truncated channel inversion formulation for the last part (again using continuous modulation assumption)

Problem 2
Use the fact that the number of constellation points is finite; maximize spectral efficiency over all possible $\gamma_0$ (dictated by the available constellations)

Problem 3
Use the Markov model in the first part and a different BER expression for the last part
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<table>
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<th>Basic effect</th>
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<td>True value $\gamma$, but our estimate is $\hat{\gamma}$; what is the error?</td>
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<th>Some expressions</th>
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<td>- Can be characterized by $p(\gamma, \hat{\gamma})$</td>
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<tr>
<td>- Approximation for $P_b$ gives</td>
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$$P_b(\gamma, \hat{\gamma}) \approx 0.2[5P_b]^{\gamma/\hat{\gamma}}$$

(depends only on $\epsilon = \hat{\gamma}/\gamma$!)

<table>
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<td>What happens if $\gamma &gt; \hat{\gamma}$? (in terms of power and BER?)</td>
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Homework 6

Problem 4 hint

The BER under the approximation we saw depends only on the statistics of $\epsilon$ (not the joint statistics of $\gamma$ and $\hat{\gamma}$)

Problem 5

- For part a, modulation used when $P_b(\gamma) \leq P_{\text{floor}}$.
- For part b, $y_t, h_t, h_{t+1}$ are jointly gaussian. $y_t = x_t h_t + n_t$. Can be used to find MMSE estimate $\hat{h}_t = E[h_t|y_t, x_t]$.
- For part c, $h_{t+1} = ah_t + \sqrt{1-a^2}n_{t,2}$ for appropriate $a$. Can be used to find MMSE estimate.
- $P_b(|h_{t+1}|^2, |\hat{h}_{t+1}|^2)$ is the probability of error of modulation scheme (chosen by estimate $|\hat{h}_{t+1}|^2$) with SNR $|\hat{h}_{t+1}|^2$. 
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Representing MIMO systems

Assumptions
- Narrowband signals
- $N_t$ transmit antennas
- $N_r$ receive antennas
- Noise $\mathbf{n}$ is zero mean with a covariance matrix of $\sigma^2 \mathbf{I}_{N_r}$

Idea
Represent gain from transmitter antenna $j$ to receiver antenna $i$ by $h_{i,j}$
Assumptions

- Narrowband signals
- $N_t$ transmit antennas
- $N_r$ receive antennas
- Noise $\mathbf{n}$ is zero mean with a covariance matrix of $\mathbf{I}_{N_r}$

Idea

Represent gain from transmitter antenna $j$ to receiver antenna $i$ by $h_{i,j}$
System model

Model

\[
\begin{bmatrix}
  y_1 \\
  \vdots \\
  y_{N_r}
\end{bmatrix} =
\begin{bmatrix}
  h_{1,1} & \cdots & h_{1,N_t} \\
  \vdots & \ddots & \vdots \\
  h_{N_r,1} & \cdots & h_{N_r,N_t}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_{N_t}
\end{bmatrix} +
\begin{bmatrix}
  n_1 \\
  \vdots \\
  n_{N_r}
\end{bmatrix}
\]

where \( x \) is what transmitter sends and \( y \) is what receiver sees.

Transmit power constraint

\[
E[xx^*] = \sum_{i=1}^{N_t} E[|x_i|^2] \leq \rho
\]
Decomposition of $H$

**Idea**
Use the singular value decomposition (SVD) of channel matrix

$$H = U \Sigma V^H$$

**Parallel channel decomposition**
- Transmitter sends $x = V \tilde{x}$ (transmit precoding)
- Receiver obtains $\tilde{y} = U^H y$ (receiver shaping)

$$\tilde{y} = \Sigma \tilde{x}$$

**Figure:** Equivalent parallel channels (no “crosstalk” or interchannel interference)
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Channel capacity with Tx and Rx CSI

**Expression**

Under system model, can be shown to be

\[
C = \max_{R_x: \text{Tr}(R_x) \leq \rho} B \log_2 \left| I + \frac{1}{\sigma^2} H R_x H^H \right|
\]

**Equivalent expression**

By using parallel decomposition, we get

\[
C = \max_{\rho: \sum_i \rho_i \leq \rho} B \sum_i \log_2 (1 + \rho_i \sigma_i^2)
\]

**Question**

How do you solve this?
Channel capacity with Tx and Rx CSI

**Expression**

Under system model, can be shown to be

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\]

**Question**

How do you solve this? Waterfill!
Channel capacity with Rx CSI only

**Idea**

Cannot use the channel realization at the transmitter; so spread energy equally at all the transmitters

**Capacity expression**

\[
C = \max_{R_x: R_x = \rho/N_t I_{N_t}} B \log_2 \left| I + H R_x H^H \right|
\]

**Equivalent expression**

\[
C = \sum_i B \log_2 (1 + \rho \sigma_i^2 / N_t)
\]
Homework 6

Problem 6
- For part a, use symmetry or AM-GM inequality
- For part b, use Hadamard’s inequality

Problem 7
Use the law of large numbers and note that $M_t$ is fixed