EE359 Discussion Session 8
Beamforming, Diversity-multiplexing tradeoff, MIMO receiver design, Multicarrier modulation

November 18, 2015
Outline

1 Recap

2 MIMO concepts
   - Beamforming
   - Diversity multiplexing tradeoff for point to point MIMO
   - MIMO receiver design

3 Multicarrier modulation
Last discussion session

- Adaptive modulation and adaptive power schemes
- Introduction to MIMO (Multiple Input, Multiple Output) systems

This session

- MIMO beamforming
- Diversity multiplexing tradeoff
- MIMO receiver design
- Multicarrier modulation
Outline

1 Recap

2 MIMO concepts
   - Beamforming
   - Diversity multiplexing tradeoff for point to point MIMO
   - MIMO receiver design

3 Multicarrier modulation
Brief recap of the notation for a point to point MIMO system

\[ y = Hx + n \equiv \tilde{y} = \Sigma \tilde{x} + \tilde{n} \]

where \( H = U\Sigma V^H = \sum_i \sigma_i u_i v_i^H \) and \( x = V\tilde{x}, \tilde{y} = U^Hy \)

- \( N_t \) transmit antennas and \( N_r \) receive antennas
- Decomposition into parallel channels with perfect CSIT and CSIR
- \( \sigma_1 > \sigma_2 > \ldots \)
Outline

1 Recap

2 MIMO concepts
   - Beamforming
   - Diversity multiplexing tradeoff for point to point MIMO
   - MIMO receiver design

3 Multicarrier modulation
Beamforming

Idea
If CSIT available, simply transmit along vector with the largest singular value, i.e. make
$$\tilde{x} \in \mathbb{C} \text{ scalar - one value}$$

Some points
- Equivalent scalar channel $$\tilde{y} = u^H H v \tilde{x} + \tilde{n}_1$$
- Maximizes SNR if $$u$$ and $$v$$ are first singular vectors
- Capacity optimal only if other parallel channels are “weak”
- Why is maximum SNR scheme not capacity optimal always?
- Any choice of $$u$$ and $$v$$ other than $$u_1$$ and $$v_1$$ is suboptimal
Problem 1
Show that optimal (i.e., SNR maximizing) choice for $u$ and $v$ are $u_1$ and $v_1$ respectively

Problem 2
Problem 2 explores linear algebra inequality between 2-norm and Frobenius norm. Connection between MIMO communication and diversity in Rayleigh fading.
Outline

1 Recap

2 MIMO concepts
   - Beamforming
   - Diversity multiplexing tradeoff for point to point MIMO
   - MIMO receiver design

3 Multicarrier modulation
The tradeoff

Setting
CSI (channel state info) known at receiver but is unknown at transmitter, finite blocklengths.

Intuition
Antennas can be used for higher reliability (diversity) or rate (multiplexing), but not both simultaneously

Fineprint
- We assume i.i.d. complex normal entries for $H$
- High SNR concept:
  - Multiplexing gain $r = \lim_{SNR \to \infty} \frac{R(SNR)}{\log_2(SNR)}$
  - Diversity gain $d = \lim_{SNR \to \infty} \frac{-\log P_e}{\log SNR}$
The tradeoff

- Blue dot corresponds to low rate, high reliability transmission
- Red dot corresponds to high rate, low reliability transmission

**Achievability**
Any point on this tradeoff curve may be achieved in general by a suitable space time code

**Figure:** Blue curve for $N_t = 3, N_r = 3$, green for $N_t = 2, N_r = 2$
Problem 3
Explores the DMT for a simple system

Problem 4
Evaluate capacity. Also evaluate bit error rates and total throughput for some achievable schemes (Note that DMT is a property of the channel, not of any scheme)
Outline

1 Recap

2 MIMO concepts
   - Beamforming
   - Diversity multiplexing tradeoff for point to point MIMO
   - MIMO receiver design

3 Multicarrier modulation
The optimal receiver

Idea

Maximum likelihood criterion: return $\hat{x} = \arg\max_{x \in X} f(H, y, x)$

Some more details about ML decoder

- Minimum probability of error if $x$ are equiprobable in $X$
- For iid Gaussian noise statistics and equiprobable BPSK signalling, $f(H, y, x) = e^{-c||y - Hx||^2}$, so

$$\hat{x} = \arg\min_{x \in \{-1, +1\}^{N_t}} ||y - Hx||^2$$

Problem

Exponential complexity in $N_t$
Towards a simpler decoder: Zero forcing

Idea
Use matrix inversion

Math
\[ \hat{x} = H^\dagger y \text{ where } H^\dagger = (H^H H)^{-1} H^H \text{ if } H \text{ is “tall”} \]

Some features
- Good (linear) complexity \(N_t^3\) if \(H\) is square
- If singular values are unequal (i.e. matrix is ill-conditioned), noise enhancement

Towards a simpler decoder: Linear MMSE decoding

**Idea**
- Write estimate as an affine function of $y$. Minimize expected squared error by choosing right affine function.
- Regular MMSE: Assume $x$ to be i.i.d. multivariate Gaussian and compute optimal decoder (minimum expected mean squared error (MSE)).

**Math (assuming SNR = $1/\sigma^2$)**

$$\hat{x} = (H^H H + \sigma^2 I)^{-1} H^H y$$

**Some features**
- Good complexity (similar to zero forcing)
- Less sensitive to ill-conditioned matrices
- In practice $x$ is not Gaussian
Near ML decoding: Sphere decoder

Idea
Use the $\mathbf{Q}\mathbf{R}$ decomposition of $\mathbf{H}$ ($\mathbf{R}$ upper triangular, $\mathbf{Q}$ unitary)

Math
$$||\mathbf{y} - \mathbf{H}\mathbf{x}||^2 = ||\mathbf{Q}^\mathbf{H}\mathbf{y} - \mathbf{R}\mathbf{x}||^2 = \sum_i ((\mathbf{Q}^\mathbf{H}\mathbf{y})_i - \sum_{j \geq i} R_{i,j}x_j)^2$$

Figure: Tree with $2^{N_t}$ possible paths (for BPSK)

Algorithm (depth first search)
- Traverse tree depth first
- Prune branches of tree if accumulated sum is greater than $r$
- Smaller $r$ leads to lower complexity but possibly higher BER
Near ML decoding: Sphere decoder

Idea
Use the QR decomposition of $\mathbf{H}$ ($\mathbf{R}$ upper triangular, $\mathbf{Q}$ unitary)

Math
$$||\mathbf{y} - \mathbf{Hx}||^2 = ||\mathbf{Q}^\mathbf{H}\mathbf{y} - \mathbf{Rx}||^2 = \sum_i ((\mathbf{Q}^\mathbf{H}\mathbf{y})_i - \sum_{j \geq i} R_{i,j} x_j)^2$$

![Figure: Tree with $2^{N_t}$ possible paths (for BPSK)]

Algorithm (depth first search)
- Traverse tree depth first
- Prune branches of tree if accumulated sum is greater than $r$
- Smaller $r$ leads to lower complexity but possibly higher BER
Homework 7

Problem 5
Uses concepts discussed so far

Problem 6
With normal noise, ML decoder returns closest point. Sphere decoder returns closest point if possible.
Outline

1 Recap

2 MIMO concepts
   - Beamforming
   - Diversity multiplexing tradeoff for point to point MIMO
   - MIMO receiver design

3 Multicarrier modulation
## Problem
Coherence bandwidth of channel is small, thus channel “spreads” wideband signal in time

## Some common remedies
- Equalization/deconvolution/channel inversion
- Multicarrier modulation
- Spread spectrum
## Multicarrier modulation

### Idea

Split wideband ($B$) into $N$ narrowband chunks each of bandwidth $B/N$, such that

$$B_n = \frac{B}{N} \leq B_c$$

### Common incarnations

- Frequency division multiplexing (FDM)
- Orthogonal FDM (OFDM)

### Uses

- 4G LTE, Wifi use OFDM
- 2G standards (GSM) used FDM heavily
FDM

Idea
Pack a bunch of orthonormal basis functions in frequency domain, thereby creating parallel channels

Implementation issues
- Minimum carrier frequency separation with signal duration $T_N$ is $1/T_N$
- Usually need a rolloff factor $\beta$ and guard bands $\epsilon$, thus effective occupancy $B_n = N(1 + \beta + \epsilon)/T_N$
- Need separate receiver hardware/modulation schemes at each carrier frequency
Problem 7

Two signals $s_i(t)$ and $s_j(t)$ over time $T_N$ are orthogonal if

$$\int_{t=0}^{T_N} s_i(t)s_j(t)dt = 0$$