1. (10 pts) **Successive decoding** (Problem 4-3): Consider two users simultaneously transmitting to a single receiver in an AWGN channel. This is a typical scenario in a cellular system with multiple users sending signals to a base station. Assume the users have equal received power of 10 mW and total noise at the receiver in the bandwidth of interest of 0.1 mW. The channel bandwidth for each user is 20 MHz.

   (a) (5 pts) Suppose that the receiver decodes user 1’s signal first. In this decoding, user 2’s signal acts as noise (assume it has the same statistics as AWGN). What is the capacity of user 1’s channel with this additional interference noise?

   (b) (5 pts) Suppose that, after decoding user 1’s signal, the decoder re-encodes it and subtracts it out of the received signal. Now, in the decoding of user 2’s signal, there is no interference from user 1’s signal. What then is the Shannon capacity of user 2’s channel?

   Note: It is shown in Chapter 14 that the decoding strategy of successively subtracting out decoded signals is optimal for achieving Shannon capacity of a multiuser channel with independent transmitters sending to one receiver.

2. (10 pts) **Capacity with only CSIR** (Problem 4-4): Consider a flat fading channel of bandwidth 20 MHz and where, for a fixed transmit power $\bar{P}$, the received SNR is one of six values: $\gamma_1 = 20$ dB, $\gamma_2 = 15$ dB, $\gamma_3 = 10$ dB, $\gamma_4 = 5$ dB, $\gamma_5 = 0$ dB, and $\gamma_6 = -5$ dB. The probabilities associated with each state are $p_1 = p_6 = 0.1, p_2 = p_4 = 0.15$, and $p_3 = p_5 = 0.25$. Assume that only the receiver has CSI.

   (a) (5 pts) Find the Shannon capacity of this channel.

   (b) (5 pts) For this part, please review Section 4.2.3 in the book about capacity vs. outage under RX CSI only (not covered in class). Plot the capacity versus outage for $0 \leq P_{\text{out}} < 1$ and find the maximum average rate that can be correctly received (maximum $C_{\text{out}}$).

3. (15 pts) **Rayleigh fading and outage capacity**: In this problem we explore some properties of Rayleigh fading (with average received power 1, assuming a transmit power of 1). In particular, we investigate why we should not use channel inversion when the instantaneous gain $\gamma$ is really low. In the following questions, we use the following fact for the Rayleigh pdf

   $$p(\gamma) = e^{-\gamma} > 1 - \gamma,$$

   where the right hand is the first order Taylor series expansion of $e^{-\gamma}$ around $\gamma = 0$.

   (a) (4 pts) For a small positive constant $c$ and a smaller positive constant $\epsilon < c$, compute a lower bound for $\mathbb{E} \left[ \frac{1}{\gamma} \right]$ by integrating the above lower bound within the interval $\gamma \in [\epsilon, c]$. This is a lower bound on the transmit power needed to maintain a constant received power.
(b) (3 pts) As $\epsilon \to 0$, show that the above expression is unbounded. This suggests that the transmit power needed to maintain a constant received power is unbounded.

(c) (6 pts) Consider a probability density function $p(\gamma)$ whose value for $\gamma \in (0, c)$ is $\gamma^l$ for a real number $l > -1$. Note that Rayleigh fading corresponds roughly to $l = 0$. For what values of $l$ is $E[1/\gamma]$ unbounded?

Hint: Consider the same approach as in the previous part, i.e. compute $\int_c^\epsilon \frac{1}{\gamma} p(\gamma) d\gamma$, and let $\epsilon \to 0$.

(d) (2 pts) Does your answer in the above question depend on the value of the probability density function over $\gamma \in (c, \infty)$?

Hint: Note that $1/\gamma$ in $(c, \infty)$ can be upper bounded by the constant $1/c$.

4. (15 pts) **Optimal and suboptimal power adaptation** (Problem 4-6):

Consider a cellular system where the power falloff with distance follows the formula $P_r(d) = P_t\left(\frac{d_0}{d}\right)^\alpha$, where $d_0 = 100$ m and $\alpha$ is a random variable. The distribution for $\alpha$ is $p(\alpha = 2) = .4, p(\alpha = 2.5) = .3, p(\alpha = 3) = .2, \text{ and } p(\alpha = 4) = .1$. Assume a receiver at a distance $d = 1000$ m from the transmitter, with an average transmit power constraint of $P_t = 100$ mW and a receiver noise power of $.1$ mW. Assume that both transmitter and receiver have CSI.

(a) (2 pts) Compute the distribution of the received SNR.

(b) (5 pts) Derive the optimal power adaptation policy for this channel and its corresponding Shannon capacity per unit hertz ($C/B$).

(c) (3 pts) Determine the zero-outage capacity per unit bandwidth of this channel.

(d) (5 pts) Determine the maximum outage capacity per unit bandwidth of this channel.

5. (15 pts) **More on shannon capacity** (Problem 4-7): Assume a Rayleigh fading channel, where the transmitter and receiver have CSI and the distribution of the fading SNR $p(\gamma)$ is exponential with mean $\bar{\gamma} = 10$ dB. Assume a channel bandwidth of 10 MHz.

(a) Find the cutoff value $\gamma_0$ and the corresponding power adaptation that achieves Shannon capacity on this channel.

(b) Compute the Shannon capacity of this channel.

(c) Compare your answer in part (b) with the channel capacity in AWGN with the same average SNR.

(d) Compare your answer in part (b) with the Shannon capacity when only the receiver knows $\gamma[i]$.

(e) Compare your answer in part (b) with the zero-outage capacity and outage capacity when the outage probability is .05.

(f) Repeat parts (b), (c), and (d) - that is, obtain the Shannon capacity with perfect transmitter and receiver side information, in AWGN for the same average power, and with just receiver side information - for the same fading distribution but with mean $\bar{\gamma} = -5$ dB. Describe the circumstances under which a fading channel has higher capacity than an AWGN channel with the same average SNR and explain why this behavior occurs.

6. (25 pts) **Capacity with interference** (Problems 4-8 & 4-9): This problem illustrates the capacity gains that can be obtained from interference estimation and also how a malicious jammer can wreak havoc on link performance. Consider the interference channel depicted in Figure 1. The channel has a combination of AWGN $n[k]$ and interference $I[k]$. We model $I[k]$ as AWGN. The interferer is on (i.e., the switch is down) with probability .25 and off (i.e., switch up) with probability .75. The average transmit power is 10 mW, the noise PSD has $N_0 = 10^{-8}$ W/Hz, the channel bandwidth $B$ is 10 kHz (receiver noise power is $N_0 B$), and the interference power (when on) is 9 mW.
Figure 1: Figure for problem 6

(a) (5 pts) What is the Shannon capacity of the channel if neither transmitter nor receiver know when the interferer is on (the interference power is known however)?

(b) (5 pts) What is the capacity of the channel if both transmitter and receiver know when the interferer is on?

(c) (5 pts) Suppose now that the interferer is a malicious jammer with perfect knowledge of $x[k]$ (so the interferer is no longer modeled as AWGN). Assume that neither transmitter nor receiver has knowledge of the jammer behavior. Assume also that the jammer is always on and has an average transmit power of 10 mW. What strategy should the jammer use to minimize the SNR of the received signal?

(d) (10 pts) Consider the malicious interferer from the previous problem. Suppose that the transmitter knows the interference signal perfectly. Consider two possible transmit strategies under this scenario: the transmitter can ignore the interference and use all its power for sending its signal, or it can use some of its power to cancel out the interferer (i.e. transmit the negative of the interference signal). In the first approach the interferer will degrade capacity by increasing the noise, and in the second strategy the interferer also degrades capacity since the transmitter sacrifices some power to cancel out the interference. Which strategy results in higher capacity? Note: there is a third strategy, where the encoder actually exploits the structure of the interference in its encoding. This strategy is called dirty paper coding, and is used to achieve Shannon capacity on broadcast channels with multiple antennas.

7. (10 pts) Waterfilling formula derivation (Problem 4-10): Show using Lagrangian techniques that the optimal power allocation to maximize the capacity of a time-invariant block fading channel is given by the water-filling formula in Equation (4.24).