EE359 Discussion Session 2
Statistical Fading Models

October 7, 2014
Announcements/reminders/clarifications

- If you haven’t already, please let us know which slots work for you for the midterm (on Piazza)
- Up to 3 people can turn in one homework writeup
Outline

1. Time varying channel

2. Towards a statistical characterization

3. Signal envelope distributions
Outline

1. Time varying channel

2. Towards a statistical characterization

3. Signal envelope distributions
The time varying channel model

Basic idea

The channel is linear but may *not* be time invariant

Modelling linear time varying channel

\[ r(t) = Re \left\{ \left( \int_{-\infty}^{\infty} c(\tau) u(t - \tau) d\tau \right) e^{j2\pi f_c t} \right\} \]
The time varying channel model

Basic idea
The channel is linear but may \textit{not} be time invariant

Modelling linear time varying channel

\[ r(t) = Re \left\{ \left( \int_{-\infty}^{\infty} c(\tau, t)u(t - \tau)d\tau \right) e^{j2\pi f_c t} \right\} \]

- \( c(\tau, t) \) is channel response at time \( t \) to an impulse at time \( t - \tau \)
- Multipath model is when

\[ c(\tau, t) = \sum_{n=1}^{N(t)} \alpha_n(t) e^{-j\phi_n(t)} \delta(\tau - \tau_n(t)) \]
Multipath model in more detail

Received signal

\[ r(t) = \text{Re} \left\{ \left( \sum_{n=1}^{N(t)} \alpha_n(t) e^{-j\phi_n(t)} u(t - \tau_n(t)) \right) e^{j2\pi f_c t} \right\} \]

- \( N(t) \) is possibly random number of multipath components
Multipath model in more detail

Figure: Multipath channel

Received signal

\[ r(t) = \text{Re} \left\{ \left( \sum_{n=1}^{N(t)} \alpha_n(t) e^{-j\phi_n(t)} u(t - \tau_n(t)) \right) e^{j2\pi f_c t} \right\} \]

- \( \tau_n(t) \) is the delay of the \( n^{th} \) multipath component
Multipath model in more detail

\[ r(t) = \Re \left\{ \sum_{n=1}^{N(t)} \alpha_n(t) e^{-j\phi_n(t)} u(t - \tau_n(t)) e^{j2\pi f_c t} \right\} \]

- \( \phi_n(t) = 2\pi f_c \tau_n - \int_t^{t+\tau_n} 2\pi f_D(\tau) d\tau - \phi_0 \) is the phase due to time delay and doppler in the \( n^{th} \) multipath component
Homework 2

- Problems 1 and 2 deal with coverage area derivation and computation
- Problem 3 deals with explicitly writing out multipath model for a specific channel
Outline

1. Time varying channel
2. Towards a statistical characterization
3. Signal envelope distributions
Narrowband approximation

A measure of the “spread” of $\tau_n(t)$

Non random: $T_m = \max_n \tau_n(t) - \min_n \tau_n(t)$
Random: $T_m = \text{stddev}(\tau_1, \ldots, \tau_N(t))$

Idea: Narrowband assumption

If the “spread” $T_m$ is such that

$$T_m \ll T_u,$$

the signals “overlap”, i.e. $u(t - \tau_n) \approx u(t)$
Narrowband approximation

A measure of the “spread” of $\tau_n(t)$

Non random: $T_m = \max_n \tau_n(t) - \min_n \tau_n(t)$
Random: $T_m = \text{stddev}(\tau_1, ..., \tau_N(t))$

Idea: Narrowband assumption

If the “spread” $T_m$ is such that

$$T_m \ll T_u,$$

the signals “overlap”, i.e. $u(t - \tau_n) \approx u(t)$

Question

- What happens if the above assumption is not true?
- How do you express the above in terms of the signal bandwidth (instead of $T_u$)?
Some implications of narrowband assumption

- Signal suffers only scaling by a complex factor

\[
Re \left\{ u(t) \left( \sum_{n=1}^{N(t)} \alpha_n(t) e^{-j\phi_n(t)} \right) e^{j2\pi f_c t} \right\}
\]

- If the above scaling factor is \( r_I(t) + j r_Q(t) \), then the in-phase and the quadrature components are

\[
r_I(t) = \sum_{n=1}^{N(t)} \alpha_n(t) \cos(\phi_n(t))
\]

\[
r_Q(t) = -\sum_{n=1}^{N(t)} \alpha_n(t) \sin(\phi_n(t))
\]
Statistics of the in-phase and quadrature components

Idea

If $N(t)$ is large and $\alpha_n(t)$ and $\phi_n(t)$ are i.i.d. and zero mean then use Central Limit Theorem

Implications

- $r_I(t)$ and $r_Q(t)$ are zero mean Gaussian if $\phi_n(t)$ is uniformly distributed
- Fully characterised by the second moments, $E(r_I(t)^2)$ and $E(r_Q(t)^2)$
Statistics of the in-phase and quadrature components

Idea

If $N(t)$ is large and $\alpha_n(t)$ and $\phi_n(t)$ are i.i.d. and zero mean then use Central Limit Theorem

Implications

- $r_I(t)$ and $r_Q(t)$ are zero mean Gaussian if $\phi_n(t)$ is uniformly distributed
- Fully characterised by the second moments, $E(r_I(t)^2)$ and $E(r_Q(t)^2)$

Question

What about realizations of in-phase and quadrature components at different times?
A collection of random variables $x(\cdot)$ indexed by a time parameter $t$ is a random process $x(t)$. 

Specifying a general zero mean random process

To fully characterise random processes, we need joint distribution of $x(t_1), x(t_2), \ldots, x(t_n)$ for all $n$ and for all possible $\{t_i\}_{i=1}^n$. 

Specifying a Gaussian random process

We need $E[x(t_1)], E[x(t_2)], E[x(t_1)Hx(t_2)]$ for all possible $t_1, t_2$. 

Why?

Because Gaussian random process is defined to be such that $x(t_1), x(t_2), \ldots, x(t_n)$ are jointly Gaussian for all $n$ and all $\{t_i\}_{i=1}^n$. 

EE359 Discussion 2  
October 7, 2014 12 / 22
Random processes 101

**Definition**
A collection of random variables $x(\cdot)$ indexed by a time parameter $t$ is a random process $x(t)$

**Specifying a general zero mean random process**
To fully characterise random processes, we need joint distribution of $x(t_1), x(t_2), \ldots, x(t_n)$ for all $n$ and for all possible $\{t_i\}_{i=1}^n$. Because a Gaussian random process is defined to be such that $x(t_1), x(t_2), \ldots, x(t_n)$ are jointly Gaussian for all $n$ and all $\{t_i\}_{i=1}^n$. 


### Random processes 101

#### Definition
A collection of random variables $x(\cdot)$ indexed by a time parameter $t$ is a random process $x(t)$

#### Specifying a general zero mean random process
To fully characterise random processes, we need joint distribution of $x(t_1), x(t_2), \ldots, x(t_n)$ for all $n$ and for all possible $\{t_i\}_{i=1}^{n}$

#### Specifying a Gaussian random process
We need $E[x(t_1)], E[x(t_2)],$ and $E[x(t_1)^H x(t_2)]$ for all possible $t_1, t_2$
Random processes 101

Definition
A collection of random variables $x(\cdot)$ indexed by a time parameter $t$ is a random process $x(t)$

Specifying a general zero mean random process
To fully characterise random processes, we need joint distribution of $x(t_1), x(t_2), \ldots, x(t_n)$ for all $n$ and for all possible $\{t_i\}_{i=1}^n$

Specifying a Gaussian random process
We need $E[x(t_1)], E[x(t_2)], \text{ and } E[x(t_1)^H x(t_2)]$ for all possible $t_1, t_2$

Why?
Because Gaussian random process is defined to be such that $x(t_1), x(t_2), \ldots, x(t_n)$ are jointly Gaussian for all $n$ and all $\{t_i\}_{i=1}^n$
Gaussian process approximation for $r_I(t)$ and $r_Q(t)$

**Advantages**

Fully characterised by first and second moments

**Further implications**

- Uniformity of $\phi_n(t)$ in $[0, 2\pi]$ gives zero mean for $r_I(t)$ and $r_Q(t)$
- Fully specified by

\[
A_{r_I}(t, t + \tau) = E[r_I(t)r_I(t + \tau)] \\
A_{r_I,r_Q}(t, t + \tau) = E[r_Q(t)r_I(t + \tau)] \\
A_{r_Q}(t, t + \tau) = E[r_Q(t)r_Q(t + \tau)]
\]
Computing the second moments

**Assumptions**

\[ f_c \tau \gg f_D \tau, f_D t \]

- This gives

\[ E[\cos(\phi_n(t)\phi_n(t + \tau))] = E[\cos 2\pi f_{Dn} \tau] \]

\[ A_{rI}(t, t + \tau) = A_{rQ}(t, t + \tau) = \sum_n 0.5 E[\alpha_n^2] E[\cos 2\pi f_{Dn} \tau] \]

\[ A_{rI,rQ}(t) = 0 \]

- The quantities are independent of \( t \)!

Hence \( r_I(t) \) and \( r_Q(t) \) are wide-sense stationary!
Some expressions for the second moments

Jakes’ model

Uniform scattering: Direction of arrival $\theta_n$ is uniformly distributed in $[0, 2\pi]$

- Fourier transform

$$A_{r_I}(t, t + \tau) = P_r \frac{1}{2\pi} \int_0^{2\pi} \cos \left( \frac{2\pi v\tau \cos \theta}{\lambda} \right) d\theta$$

$$= P_r J_0(2\pi f_D\tau) \quad \left( f_D = \frac{v \cos \theta}{\lambda} \right)$$

- Fourier transform $S_{r_I}(f) = \frac{2P_r}{\pi f_D} \frac{1}{\sqrt{1-(f/f_D)^2}} \operatorname{rect} \left( \frac{f}{2f_D} \right)$

Some observations

Decorrelation around $v\tau \approx 0.4\lambda$

Can use $S_{r_I}(f)$ to generate $r_I(t)$ from Gaussian noise (Homework 2, Problem 7)!
Some expressions for the second moments

**Jakes’ model**

Uniform scattering: Direction of arrival $\theta_n$ is uniformly distributed in $[0, 2\pi]$

\[ A_{rI}(t, t + \tau) = P_r \frac{1}{2\pi} \int_0^{2\pi} \cos \left( \frac{2\pi v \tau \cos \theta}{\lambda} \right) d\theta \]

\[ = P_r J_0(2\pi f_D \tau) \quad \left( f_D = \frac{v \cos \theta}{\lambda} \right) \]

**Fourier transform**

\[ S_{rI}(f) = \frac{2P_r}{\pi f_D} \frac{1}{\sqrt{1-(f/f_D)^2}} \text{rect} \left( \frac{f}{2f_D} \right) \]

**Some observations**

- Decorrelation around $v\tau \approx 0.4\lambda$
- Can use $S_{rI}(f)$ to generate $r_I(t)$ from Gaussian noise (Homework 2, Problem 7)!
Decorrelation around $f_D \tau \approx 0.4$

![Figure: Correlation as a function of time $\tau$]
Outline

1. Time varying channel

2. Towards a statistical characterization

3. Signal envelope distributions
Rayleigh fading

Observation so far
Both in-phase and quadrature components are zero mean gaussian and independent (according to model)

Some implications for $Z = r_I + jr_Q$

- The amplitude $|Z|$ is Rayleigh distributed (Homework 2, Problem 4a)
  \[ |Z| \sim \frac{2|Z|}{P_r} e^{-\frac{|Z|^2}{P_r}} \]
- The phase $\angle Z$ is uniform
- Known as Rayleigh fading
Ricean fading

Difference from Rayleigh

Either the in-phase or the quadrature component has non zero mean (i.e., it has a line of sight or LOS component)

Some implications

- The amplitude \( |Z| \) follows a Ricean distribution

\[
|Z| \sim \frac{|Z|}{\sigma^2} e^{-\frac{|Z|^2 + s^2}{2\sigma^2}} I_0 \left( \frac{|Z|s}{\sigma^2} \right)
\]

where \( 2\sigma^2 \) is power in non LOS and \( s^2 \) is power in LOS

- Often specified by a \( K \) parameter where \( K = \frac{s^2}{2\sigma^2} \)
Nakagami fading

- Parameterized by received power $P_r$ and $m$, i.e.
  \[ |Z| \sim \frac{2m^m z^{2m-1}}{\Gamma(m) P_r^m} e^{-\frac{mz^2}{P_r}}, \quad m > 0.5 \]

- Useful for deriving closed form BER expressions
- $m = 1$ corresponds to Rayleigh fading
- $m = \frac{(K+1)^2}{2K+1}$ is approximately Ricean
Homework 2

- Problems 4, 5 explore some properties of the Rayleigh, Ricean and Nakagami distributions
- Problem 6 explores some properties of independent and correlated shadowing—this is the basic idea of *diversity combining* which we will revisit later on in the course
Thanks!