EE363 homework 1

1. LQR for a triple accumulator. We consider the system \( x_{t+1} = Ax_t + Bu_t, \ y_t = Cx_t, \) with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 0 & 1
\end{bmatrix}.
\]

This system has transfer function \( H(z) = (z - 1)^{-3}, \) and is called a triple accumulator, since it consists of a cascade of three accumulators. (An accumulator is the discrete-time analog of an integrator: its output is the running sum of its input.) We’ll use the LQR cost function

\[
J = \sum_{t=0}^{N-1} u_t^2 + \sum_{t=0}^{N} y_t^2,
\]

with \( N = 50. \)

(a) Find \( P_t \) (numerically), and verify that the Riccati recursion converges to a steady-state value in fewer than about 10 steps. Find the optimal time-varying state feedback gain \( K_t, \) and plot its components \((K_t)_{11}, (K_t)_{12}, \) and \((K_t)_{13}, \) versus \( t. \)

(b) Find the initial condition \( x_0, \) with norm not exceeding one, that maximizes the optimal value of \( J. \) Plot the optimal \( u \) and resulting \( x \) for this initial condition.

2. Linear quadratic state tracking. We consider the system \( x_{t+1} = Ax_t + Bu_t. \) In the conventional LQR problem the goal is to make both the state and the input small. In this problem we study a generalization in which we want the state to follow a desired (possibly nonzero) trajectory as closely as possible. To do this we penalize the deviations of the state from the desired trajectory, i.e., \( x_t - x^d_t, \) using the following cost function:

\[
J = \sum_{\tau=0}^{N} (x_\tau - x^d_\tau)^T Q (x_\tau - x^d_\tau) + \sum_{\tau=0}^{N-1} u^T_\tau Ru_\tau,
\]

where we assume \( Q = Q^T \geq 0 \) and \( R = R^T > 0. \) (The desired trajectory \( x^d_\tau \) is given.) Compared with the standard LQR objective, we have an extra linear term (in \( x \)) and a constant term.

In this problem you will use dynamic programming to show that the cost-to-go function \( V_t(z) \) for this problem has the form

\[
z^T P_t z + 2q^T_t z + r_t,
\]

with \( P_t = P^T_t \geq 0. \) (i.e., it has quadratic, linear, and constant terms.)

(a) Show that \( V_N(z) \) has the given form.
(b) Assuming $V_{t+1}(z)$ has the given form, show that the optimal input at time $t$ can be written as

$$u_t^* = K_t x_t + g_t,$$

where

$$K_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A, \quad g_t = -(R + B^T P_{t+1} B)^{-1} B^T q_{t+1}.$$

In other words, $u_t^*$ is an affine (linear plus constant) function of the state $x_t$.

(c) Use backward induction to show that $V_0(z), \ldots, V_N(z)$ all have the given form. Verify that

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A,$$

$$q_t = (A + BK_t)^T q_{t+1} - Q x_t^d,$$

$$r_t = r_{t+1} + x_t^d Q x_t^d + q_{t+1}^T B g_t,$$

for $t = 0, \ldots, N - 1$.

3. The Schur complement. In this problem you will show that if we minimize a positive semidefinite quadratic form over some of its variables, the result is a positive semidefinite quadratic form in the remaining variables. Specifically, let

$$J(u, z) = \begin{bmatrix} u \\ z \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix}$$

be a positive semidefinite quadratic form in $u$ and $z$. You may assume $Q_{11} > 0$ and $Q_{11}, Q_{22}$ are symmetric. Define $V(z) = \min_u J(u, z)$. Show that $V(z) = z^T P z$, where $P$ is symmetric positive semidefinite (find $P$ explicitly).

The matrix $P$ is called the Schur complement of the matrix $Q_{11}$ in the big matrix above. It comes up in many contexts.

4. A useful determinant identity. Suppose $X \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{m \times n}$.

(a) Show that $\det(I + XY) = \det(I + YX)$. Hint: Find a block lower triangular matrix $L$ for which

$$\begin{bmatrix} I & X \\ -Y & I \end{bmatrix} = L \begin{bmatrix} I & X \\ 0 & I \end{bmatrix},$$

and use this factorization to evaluate the determinant of this matrix. Then find a block upper triangular matrix $U$ for which

$$\begin{bmatrix} I & X \\ -Y & I \end{bmatrix} = U \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix},$$

and repeat.
(b) Show that the nonzero eigenvalues of \(XY\) and \(YX\) are exactly the same.

5. When does a finite-horizon LQR problem have a time-invariant optimal state feedback gain? Consider a discrete-time LQR problem with horizon \(t = N\), with optimal input \(u(t) = K_t x(t)\). Is there a choice of \(Q_f\) (symmetric and positive semidefinite, of course) for which \(K_t\) is constant, i.e., \(K_0 = \cdots = K_{N-1}\)?