Lecture 3
Infinite horizon linear quadratic regulator

- infinite horizon LQR problem
- dynamic programming solution
- receding horizon LQR control
- closed-loop system
Infinite horizon LQR problem

discrete-time system $x_{t+1} = Ax_t + Bu_t$, $x_0 = x^{\text{init}}$

problem: choose $u_0, u_1, \ldots$ to minimize

$$J = \sum_{\tau=0}^{\infty} (x_\tau^T Q x_\tau + u_\tau^T R u_\tau)$$

with given constant state and input weight matrices

$$Q = Q^T \geq 0, \quad R = R^T > 0$$

\ldots an infinite dimensional problem
**problem:** it’s possible that \( J = \infty \) for all input sequences \( u_0, \ldots \)

\[
x_{t+1} = 2x_t + 0u_t, \quad x^{\text{init}} = 1
\]

let’s assume \((A, B)\) is controllable

then for any \( x^{\text{init}} \) there’s an input sequence

\[
u_0, \ldots, u_{n-1}, 0, 0, \ldots
\]

that steers \( x \) to zero at \( t = n \), and keeps it there

for this \( u \), \( J < \infty \)

and therefore, \( \min_u J < \infty \) for any \( x^{\text{init}} \)
define **value function** $V: \mathbb{R}^n \rightarrow \mathbb{R}$

$$V(z) = \min_{u_0, \ldots} \sum_{\tau=0}^{\infty} \left( x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right)$$

subject to $x_0 = z$, $x_{\tau+1} = A x_\tau + B u_\tau$

• $V(z)$ is the minimum LQR cost-to-go, starting from state $z$

• doesn’t depend on time-to-go, which is always $\infty$; infinite horizon problem is *shift invariant*
**Hamilton-Jacobi equation**

**fact:** $V$ is quadratic, *i.e.*, $V(z) = z^T P z$, where $P = P^T \geq 0$
(can be argued directly from first principles)

**HJ equation:**

\[
V(z) = \min_w (z^T Q z + w^T R w + V(Az + Bw))
\]

or

\[
z^T P z = \min_w (z^T Q z + w^T R w + (Az + Bw)^T P(Az + Bw))
\]

minimizing $w$ is $w^* = -(R + B^T P B)^{-1} B^T P A z$

so HJ equation is

\[
z^T P z = z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P(Az + Bw^*)
\]

\[
= z^T (Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A) z
\]
this must hold for all $z$, so we conclude that $P$ satisfies the ARE

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

and the optimal input is constant state feedback $u_t = Kx_t$,

$$K = -(R + B^T PB)^{-1} B^T PA$$

compared to finite-horizon LQR problem,

- value function and optimal state feedback gains are time-invariant
- we don't have a recursion to compute $P$; we only have the ARE
**fact:** the ARE has only one positive semidefinite solution $P$

i.e., ARE plus $P = P^T \geq 0$ uniquely characterizes value function

consequence: the Riccati recursion

$$P_{k+1} = Q + A^T P_k A - A^T P_k B (R + B^T P_k B)^{-1} B^T P_k A,$$  

$P_1 = Q$

converges to the unique PSD solution of the ARE  
(when $(A, B)$ controllable)

(later we’ll see direct methods to solve ARE)

thus, infinite-horizon LQR optimal control is same as steady-state finite horizon optimal control
Receding-horizon LQR control

consider cost function

\[ J_t(u_t, \ldots, u_{t+T-1}) = \sum_{\tau=t}^{\tau=t+T} (x_{\tau}^T Q x_{\tau} + u_{\tau}^T R u_{\tau}) \]

- \( T \) is called *horizon*
- same as infinite horizon LQR cost, truncated after \( T \) steps into future

if \( (u_t^*, \ldots, u_{t+T-1}^*) \) minimizes \( J_t \), \( u_t^* \) is called (\( T \)-step ahead) *optimal receding horizon control*

in words:

- at time \( t \), find input sequence that minimizes \( T \)-step-ahead LQR cost, starting at current time
- then use only the first input
example: 1-step ahead receding horizon control

find $u_t, u_{t+1}$ that minimize

$$J_t = x_t^T Q x_t + x_{t+1}^T Q x_{t+1} + u_t^T R u_t + u_{t+1}^T R u_{t+1}$$

first term doesn’t matter; optimal choice for $u_{t+1}$ is 0; optimal $u_t$ minimizes

$$x_{t+1}^T Q x_{t+1} + u_t^T R u_t = (A x_t + B u_t)^T Q (A x_t + B u_t) + u_t^T R u_t$$

thus, 1-step ahead receding horizon optimal input is

$$u_t = -(R + B^T Q B)^{-1} B^T Q A x_t$$

... a constant state feedback
in general, optimal $T$-step ahead LQR control is

$$u_t = K_T x_t, \quad K_T = -(R + B^T P_T B)^{-1} B^T P_T A$$

where

$$P_1 = Q, \quad P_{i+1} = Q + A^T P_i A - A^T P_i B (R + B^T P_i B)^{-1} B^T P_i A$$

i.e.: same as the optimal finite horizon LQR control, $T - 1$ steps before the horizon $N$

- a constant state feedback
- state feedback gain converges to infinite horizon optimal as horizon becomes long (assuming controllability)
Closed-loop system

suppose $K$ is LQR-optimal state feedback gain

$$x_{t+1} = Ax_t + Bu_t = (A + BK)x_t$$

is called **closed-loop system**

($x_{t+1} = Ax_t$ is called **open-loop system**)

is closed-loop system stable? consider

$$x_{t+1} = 2x_t + u_t, \quad Q = 0, \quad R = 1$$

optimal control is $u_t = 0x_t$, i.e., closed-loop system is unstable

**fact:** if $(Q, A)$ observable and $(A, B)$ controllable, then closed-loop system is stable