Lecture 12
Basic Lyapunov theory

- stability
- positive definite functions
- global Lyapunov stability theorems
- Lasalle’s theorem
- converse Lyapunov theorems
- finding Lyapunov functions
Some stability definitions

we consider nonlinear time-invariant system \( \dot{x} = f(x) \), where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \)

a point \( x_e \in \mathbb{R}^n \) is an \textit{equilibrium point} of the system if \( f(x_e) = 0 \)

\( x_e \) is an equilibrium point \iff \( x(t) = x_e \) is a trajectory

suppose \( x_e \) is an equilibrium point

- system is \textit{globally asymptotically stable} (G.A.S.) if for every trajectory \( x(t) \), we have \( x(t) \rightarrow x_e \) as \( t \rightarrow \infty \)
  
  (implies \( x_e \) is the unique equilibrium point)

- system is \textit{locally asymptotically stable} (L.A.S.) near or at \( x_e \) if there is an \( R > 0 \) s.t. \( \|x(0) - x_e\| \leq R \implies x(t) \rightarrow x_e \) as \( t \rightarrow \infty \)
• often we change coordinates so that \( x_e = 0 \) (\( i.e. \), we use \( \tilde{x} = x - x_e \))

• a linear system \( \dot{x} = Ax \) is G.A.S. (with \( x_e = 0 \)) ⇔ \( \Re \lambda_i(A) < 0 \), \( i = 1, \ldots, n \)

• a linear system \( \dot{x} = Ax \) is L.A.S. (near \( x_e = 0 \)) ⇔ \( \Re \lambda_i(A) < 0 \), \( i = 1, \ldots, n \)
  (so for linear systems, L.A.S. ⇔ G.A.S.)

• there are many other variants on stability (\( e.g. \), stability, uniform stability, exponential stability, \ldots)

• when \( f \) is nonlinear, establishing any kind of stability is usually very difficult
Energy and dissipation functions

consider nonlinear system \( \dot{x} = f(x) \), and function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \)

we define \( \dot{V} : \mathbb{R}^n \rightarrow \mathbb{R} \) as \( \dot{V}(z) = \nabla V(z)^T f(z) \)

\( \dot{V}(z) \) gives \( \frac{d}{dt} V(x(t)) \) when \( z = x(t) \), \( \dot{x} = f(x) \)

we can think of \( V \) as generalized energy function, and \( -\dot{V} \) as the associated generalized dissipation function
Positive definite functions

A function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) is positive definite (PD) if

- \( V(z) \geq 0 \) for all \( z \)
- \( V(z) = 0 \) if and only if \( z = 0 \)
- all sublevel sets of \( V \) are bounded

The last condition is equivalent to \( V(z) \rightarrow \infty \) as \( z \rightarrow \infty \)

Example: \( V(z) = z^T P z \), with \( P = P^T \), is PD if and only if \( P > 0 \)
Lyapunov theory

Lyapunov theory is used to make conclusions about trajectories of a system \( \dot{x} = f(x) \) (e.g., G.A.S.) without finding the trajectories (i.e., solving the differential equation)

a typical Lyapunov theorem has the form:

- **if** there exists a function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) that satisfies some conditions on \( V \) and \( \dot{V} \)

- **then**, trajectories of system satisfy some property

if such a function \( V \) exists we call it a *Lyapunov function* (that proves the property holds for the trajectories)

Lyapunov function \( V \) can be thought of as *generalized energy function* for system
A Lyapunov boundedness theorem

suppose there is a function $V$ that satisfies

- all sublevel sets of $V$ are bounded
- $\dot{V}(z) \leq 0$ for all $z$

then, all trajectories are bounded, i.e., for each trajectory $x$ there is an $R$ such that $\|x(t)\| \leq R$ for all $t \geq 0$

in this case, $V$ is called a Lyapunov function (for the system) that proves the trajectories are bounded
to prove it, we note that for any trajectory $x$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) \, d\tau \leq V(x(0))$$

so the whole trajectory lies in $\{ z \mid V(z) \leq V(x(0)) \}$, which is bounded.

also shows: every sublevel set $\{ z \mid V(z) \leq a \}$ is invariant.
A Lyapunov global asymptotic stability theorem

suppose there is a function \( V \) such that

- \( V \) is positive definite
- \( \dot{V}(z) < 0 \) for all \( z \neq 0 \), \( \dot{V}(0) = 0 \)

then, every trajectory of \( \dot{x} = f(x) \) converges to zero as \( t \to \infty \) (i.e., the system is globally asymptotically stable)

interpretation:

- \( V \) is positive definite generalized energy function
- energy is always dissipated, except at 0
Proof

suppose trajectory \( x(t) \) does not converge to zero.

\( V(x(t)) \) is decreasing and nonnegative, so it converges to, say, \( \epsilon \) as \( t \to \infty \).

Since \( x(t) \) doesn’t converge to 0, we must have \( \epsilon > 0 \), so for all \( t \),
\[ \epsilon \leq V(x(t)) \leq V(x(0)). \]

\( C = \{ z \mid \epsilon \leq V(z) \leq V(x(0)) \} \) is closed and bounded, hence compact. So \( \dot{V} \) (assumed continuous) attains its supremum on \( C \), i.e., \( \sup_{z \in C} \dot{V} = -a < 0 \). Since \( \dot{V}(x(t)) \leq -a \) for all \( t \), we have

\[ V(x(T)) = V(x(0)) + \int_0^T \dot{V}(x(t)) \, dt \leq V(x(0)) - aT \]

which for \( T > V(x(0))/a \) implies \( V(x(0)) < 0 \), a contradiction.

So every trajectory \( x(t) \) converges to 0, i.e., \( \dot{x} = f(x) \) is G.A.S.
A Lyapunov exponential stability theorem

suppose there is a function $V$ and constant $\alpha > 0$ such that

- $V$ is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all $z$

then, there is an $M$ such that every trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq Me^{-\alpha t/2}\|x(0)\|$$

(this is called global exponential stability (G.E.S.))

**idea:** $\dot{V} \leq -\alpha V$ gives guaranteed minimum dissipation rate, proportional to energy
Example

consider system

\[
\dot{x}_1 = -x_1 + g(x_2), \quad \dot{x}_2 = -x_2 + h(x_1)
\]

where \(|g(u)| \leq |u|/2, \ |h(u)| \leq |u|/2\)

two first order systems with nonlinear cross-coupling
let’s use Lyapunov theorem to show it’s globally asymptotically stable

we use \( V = \left( x_1^2 + x_2^2 \right) / 2 \)

required properties of \( V \) are clear (\( V \geq 0 \), etc.)

let’s bound \( \dot{V} \):

\[
\begin{align*}
\dot{V} & = x_1 \dot{x}_1 + x_2 \dot{x}_2 \\
& = -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1) \\
& \leq -x_1^2 - x_2^2 + |x_1 x_2| \\
& \leq -(1/2)(x_1^2 + x_2^2) \\
& = -V
\end{align*}
\]

where we use \(|x_1 x_2| \leq (1/2)(x_1^2 + x_2^2) \) (derived from \((|x_1| - |x_2|)^2 \geq 0\))

we conclude system is G.A.S. (in fact, G.E.S.)

\( \textit{without knowing the trajectories} \)
Lasalle’s theorem

Lasalle’s theorem (1960) allows us to conclude G.A.S. of a system with only $\dot{V} \leq 0$, along with an observability type condition

we consider $\dot{x} = f(x)$

suppose there is a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $V$ is positive definite
- $\dot{V}(z) \leq 0$
- the only solution of $\dot{w} = f(w), \dot{V}(w) = 0$ is $w(t) = 0$ for all $t$

then, the system $\dot{x} = f(x)$ is G.A.S.
• last condition means no nonzero trajectory can hide in the “zero dissipation” set

• unlike most other Lyapunov theorems, which extend to time-varying systems, Lasalle’s theorem requires time-invariance
A Lyapunov instability theorem

suppose there is a function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) such that

- \( \dot{V}(z) \leq 0 \) for all \( z \) (or just whenever \( V(z) \leq 0 \))
- there is \( w \) such that \( V(w) < V(0) \)

then, the trajectory of \( \dot{x} = f(x) \) with \( x(0) = w \) does not converge to zero (and therefore, the system is not G.A.S.)

to show it, we note that \( V(x(t)) \leq V(x(0)) = V(w) < V(0) \) for all \( t \geq 0 \)

but if \( x(t) \rightarrow 0 \), then \( V(x(t)) \rightarrow V(0) \); so we cannot have \( x(t) \rightarrow 0 \)
A Lyapunov divergence theorem

suppose there is a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $\dot{V}(z) < 0$ whenever $V(z) < 0$
- there is $w$ such that $V(w) < 0$

then, the trajectory of $\dot{x} = f(x)$ with $x(0) = w$ is unbounded, i.e.,

$$\sup_{t \geq 0} \|x(t)\| = \infty$$

(this is not quite the same as $\lim_{t \rightarrow \infty} \|x(t)\| = \infty$)
Proof of Lyapunov divergence theorem

Let $\dot{x} = f(x)$, $x(0) = w$. Let's first show that $V(x(t)) \leq V(w)$ for all $t \geq 0$.

If not, let $T$ denote the smallest positive time for which $V(x(T)) = V(w)$. Then over $[0, T]$, we have $V(x(t)) \leq V(w) < 0$, so $\dot{V}(x(t)) < 0$, and so

$$\int_0^T \dot{V}(x(t)) \, dt < 0$$

the lefthand side is also equal to

$$\int_0^T \dot{V}(x(t)) \, dt = V(x(T)) - V(x(0)) = 0$$

so we have a contradiction.

It follows that $V(x(t)) \leq V(x(0))$ for all $t$, and therefore $\dot{V}(x(t)) < 0$ for all $t$.

Now suppose that $\|x(t)\| \leq R$, i.e., the trajectory is bounded.

$\{z \mid V(z) \leq V(x(0)), \|z\| \leq R\}$ is compact, so there is a $\beta > 0$ such that $\dot{V}(z) \leq -\beta$ whenever $V(z) \leq V(x(0))$ and $\|z\| \leq R$. 
we conclude $V(x(t)) \leq V(x(0)) - \beta t$ for all $t \geq 0$, so $V(x(t)) \to -\infty$, a contradiction.
Converse Lyapunov theorems

a typical *converse Lyapunov theorem* has the form

- **if** the trajectories of system satisfy some property
- **then** there exists a Lyapunov function that proves it

a sharper converse Lyapunov theorem is more specific about the form of the Lyapunov function

*example:* if the linear system $\dot{x} = Ax$ is G.A.S., then there is a quadratic Lyapunov function that proves it (we’ll prove this later)
A converse Lyapunov G.E.S. theorem

suppose there is $\beta > 0$ and $M$ such that each trajectory of $\dot{x} = f(x)$ satisfies
\[ \|x(t)\| \leq Me^{-\beta t}\|x(0)\| \text{ for all } t \geq 0 \]
(called \textit{global exponential stability}, and is stronger than G.A.S.)

then, there is a Lyapunov function that proves the system is exponentially stable, \textit{i.e.}, there is a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and constant $\alpha > 0$ s.t.

- $V$ is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all $z$
Proof of converse G.E.S. Lyapunov theorem

suppose the hypotheses hold, and define

\[ V(z) = \int_{0}^{\infty} \|x(t)\|^2 \, dt \]

where \( x(0) = z, \dot{x} = f(x) \)

since \( \|x(t)\| \leq Me^{-\beta t} \|z\| \), we have

\[ V(z) = \int_{0}^{\infty} \|x(t)\|^2 \, dt \leq \int_{0}^{\infty} M^2 e^{-2\beta t} \|z\|^2 \, dt = \frac{M^2}{2\beta} \|z\|^2 \]

(which shows integral is finite)
let's find $\dot{V}(z) = \left. \frac{d}{dt} \right|_{t=0} V(x(t))$, where $x(t)$ is trajectory with $x(0) = z$

\[
\dot{V}(z) = \lim_{t \to 0} \frac{1}{t} (V(x(t)) - V(x(0))) \\
= \lim_{t \to 0} \frac{1}{t} \left( \int_{t}^{\infty} \|x(\tau)\|^2 \, d\tau - \int_{0}^{\infty} \|x(\tau)\|^2 \, d\tau \right) \\
= \lim_{t \to 0} \frac{-1}{t} \int_{0}^{t} \|x(\tau)\|^2 \, d\tau \\
= -\|z\|^2
\]

now let's verify properties of $V$

$V(z) \geq 0$ and $V(z) = 0 \iff z = 0$ are clear

finally, we have $\dot{V}(z) = -z^T z \leq -\alpha V(z)$, with $\alpha = 2\beta/M^2$
Finding Lyapunov functions

- there are many different types of Lyapunov theorems
- the key in all cases is to *find* a Lyapunov function and verify that it has the required properties
- there are several approaches to finding Lyapunov functions and verifying the properties

one common approach:

- decide form of Lyapunov function (*e.g.*, quadratic), parametrized by some parameters (called a *Lyapunov function candidate*)
- try to find values of parameters so that the required hypotheses hold
Other sources of Lyapunov functions

- value function of a related optimal control problem
- linear-quadratic Lyapunov theory (next lecture)
- computational methods
- converse Lyapunov theorems
- graphical methods (really!)

(as you might guess, these are all somewhat related)