Linear quadratic Lyapunov theory

Lyapunov equations

We assume $A \in \mathbb{R}^{n \times n}$, $P = P^T \in \mathbb{R}^{n \times n}$. It follows that $Q = Q^T \in \mathbb{R}^{n \times n}$.

Continuous-time linear system: for $\dot{x} = Ax$, $V(z) = z^T P z$, we have $\dot{V}(z) = -z^T Q z$, where $P$, $Q$ satisfy (continuous-time) Lyapunov equation $A^T P + PA + Q = 0$.

Discrete-time linear system: for $x(t+1) = Ax(t)$, $V(z) = z^T P z$, we have $\Delta V(z) = -z^T Q z$, where $P$, $Q$ satisfy (discrete-time) Lyapunov equation $A^T PA - P + Q = 0$.

Lyapunov theorems

- If $P > 0$, $Q > 0$, then system is (globally asymptotically) stable.
- If $P > 0$, $Q \geq 0$, and $(Q, A)$ observable, then system is (globally asymptotically) stable.
- If $P > 0$, $Q \geq 0$, then all trajectories of the system are bounded
- If $Q \geq 0$, then the sublevel sets $\{z | z^T P z \leq a\}$ are invariant. (These are ellipsoids if $P > 0$.)
- If $P \not\geq 0$ and $Q \geq 0$, then $A$ is not stable.

Converse theorems

- If $A$ is stable, there exists a quadratic Lyapunov function $V(z) = z^T P z$ that proves it, i.e., there exists $P > 0$, $Q > 0$ that satisfies the (continuous- or discrete-time) Lyapunov equation.
- If $A$ is stable and $Q \geq 0$, then $P \geq 0$.
- If $A$ is stable, $Q \geq 0$, and $(Q, A)$ observable, then $P > 0$.

Lyapunov equation solvability conditions

- The discrete-time Lyapunov equation has a unique solution $P$, for any $Q = Q^T$, if and only if $\lambda_i(A) \lambda_j(A) \neq 1$, for $i, j = 1, \ldots, n$.
- If $A$ is stable, Lyapunov equation has a unique solution $P$, for any $Q = Q^T$.  

Integral (sum) solution of Lyapunov equation

- If $\dot{x} = Ax$ is (globally asymptotically) stable and $Q = Q^T$,
  \[ P = \int_0^\infty e^{A^T t} Q e^{A t} \, dt \]
  is the unique solution of the Lyapunov equation $A^T P + PA + Q = 0$.

- If $x(t + 1) = Ax(t)$ is (globally asymptotically) stable and $Q = Q^T$,
  \[ P = \sum_{t=0}^\infty (A^T)^t Q A^t \]
  is the unique solution of the Lyapunov equation $A^T PA - P + Q = 0$. 

2