This review session summarizes our treatment of invariant subspaces, the Sylvester operator, and the PBH criterion. The material in this review session can be found in EE363 Lecture 6.

Announcements: None.

Invariant subspaces

Suppose $A$ is an $n \times n$ matrix. A subspace $V$ of $\mathbb{R}^n$ or $\mathbb{C}^n$ (if $A$ is complex) is called $A$-invariant or an invariant subspace of $A$, if for every $v \in V$ the vector $Av$ is also in $V$. A simple example is when $V$ is spanned by a single eigenvector of $A$. More generally, $V$ may be spanned by a subset of the eigenvectors of $A$.

Example: Suppose that $V$ is $A$-invariant. Show that if $R(B) \subseteq V$, then $R(C) \subseteq V$. Here $C$ is the controllability matrix associated with $A$ and $B$.

Solution: Let $v, w$ be two vectors such that $v = Cw$. Then,

$$ v = Bw_1 + ABw_2 + \cdots + A^{n-1}Bw_n $$

since $R(B) \subseteq V$ the vectors $Bw_i$ will all lie in $V$, since $V$ is $A$-invariant $A^kBw_{k+1}$ will also lie in $V$. By observing that $v$ above is a linear combination of vectors that all lie in $V$, we have that $v \subseteq V$. Since this holds for all $w$’s we get $R(C) \subseteq V$.

Example: It was shown in the notes that $R(M)$ is $A$-invariant if and only if there is an $X$ such that $AM = MX$. In addition, if this condition holds and $M \in \mathbb{R}^{n \times k}$ has rank $k$, then the eigenvalues of $X$ are a subset of the eigenvalues of $A$. Claim: each Jordan block of $X$ is a submatrix of a Jordan block of $A$. Provide a proof or counterexample.

Solution: Take any Jordan block of $X$. Suppose it is associated with eigenvalue $\lambda$ and has size $m$. Let $v_1, \ldots, v_m$ be its generalized eigenvectors, and by definition, we have $Xv_1 = \lambda v_1, Xv_2 = \lambda v_2 + v_1, \ldots, Xv_m = \lambda v_m + v_{m-1}$. Using the condition $AM = MX$, we get $A(Mv_1) = \lambda(Mv_1), A(Mv_2) = \lambda(Mv_2) + (Mv_1), \ldots, A(Mv_m) = \lambda(Mv_m) + (Mv_{m-1})$. Since $M$ is full column rank, $Mv_1, \ldots, Mv_m$ are also linear independent. Hence they are the generalized eigenvectors of $A$ with eigenvalue $\lambda$, which implies that $A$ has a Jordan block associated with $\lambda$ of size at least $m$. 
Sylvester operator

The sylvester equation is

\[ AX + XB = C, \]

where \( A, B, X, C \in \mathbb{R}^{n \times n} \). Expressing as \( S(X) = C \), we refer to \( S(X) \) as the sylvester operator.

Example  Assuming that \( A \) and \( B \) are diagonalizable, show that the eigenvalues of the Sylvester operator are \( \lambda_i + \mu_j \) for \( i, j = 1, \ldots, n \), where \( \lambda_i \) is an eigenvalue of \( A \) and \( \mu_j \) is an eigenvalue of \( B \). What are the associated eigenvectors (matrices) \( X_{ij} \)?

Solution: Use \( X = v_i w_j^T \), where \( Av_i = \lambda_i v_i \) and \( w_j^T B = \mu_j w_j^T \). This gives

\[ S(v_i w_j^T) = A(v_i w_j^T) + (v_i w_j^T)B = (\lambda_i + \mu_j)v_i w_j^T \]

So \( \lambda_i + \mu_j \) is an eigenvalue of \( S \) with eigenvector \( v_i w_j^T \).

Example: Show that the set of \( n^2 \) matrices \( X_{ij}, i, j = 1, \ldots, n \), spans \( \mathbb{R}^{n \times n} \). This means that the Sylvester operator is diagonalizable (when \( A \) and \( B \) are).

Solution: To show that the matrices \( v_i w_j^T \) span \( \mathbb{R}^{n \times n} \), we need to show that these matrices are linear independent, i.e.,

\[ \sum_{i,j=1}^{n} \alpha_{ij}v_i w_j^T = 0 \]

if and only if \( \alpha_{ij} = 0 \) for all \( i, j \). Let \( \hat{w}_k^T \) be the \( k \)th left eigenvector of \( A \) and \( \hat{v}_l \) be the \( l \)th right eigenvector of \( B \). By the orthogonality of left and right eigenvectors,

\[ \hat{w}_k^T \left( \sum_{i,j=1}^{n} \alpha_{ij}v_i w_j^T \right) \hat{v}_l = \alpha_{kl} = 0 \]

Since this holds for all \( k, l \), the eigenvectors of the Sylvester operator span \( \mathbb{R}^{n \times n} \).

Remark. This shows that, when \( A \) and \( B \) are diagonalizable, the Sylvester operator is nonsingular if and only if no eigenvalue of \( A \) is the negative of an eigenvalue of \( B \). In fact, this holds even when \( A \) and \( B \) are not diagonalizable.

PBH

The PBH controllability and observability tests give us alternative ways to test for controllability and observability.

PBH Controllability Test: \( (A, B) \) is controllable, if and only if there exists no left eigenvector of \( A \) orthogonal to the columns of \( B \).
Remarks: There are two implications to prove. In both cases, we will prove the contraposition statement.

1. If there exists a left eigenvector of $A$ orthogonal to the columns of $B$, then $(A, B)$ is uncontrollable:

Proof: Suppose that $w \in \mathbb{C}^n, w \neq 0$ is a left eigenvector of $A$, that is orthogonal to the columns of $B$. Then

$$w^T A = \lambda w^T, \quad w^T B = 0,$$

so

$$w^T [B \ AB \ldots A^{n-1}B] = 0,$$

and therefore

$$\text{Rank}([B \ AB \ldots A^{n-1}B]) < n,$$

which means the controllability matrix has linearly dependent rows, and is not controllable.

Example: Let $\dot{x}_t = Ax_t + Bu_t$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}.$$

Is $(A, B)$ controllable?

Solution: The eigenvalues of $A$ are found to be $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$, associated with left eigenvectors

$$w_1 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

We find that $w_3^T B = 0$. In addition,

$$w_3^T \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Hence $(A, B)$ is not controllable.

2. If $(A, B)$ is not controllable, there exists a left eigenvector of $A$ orthogonal to the columns of $B$.

Proof: Suppose

$$\text{Rank}([B \ AB \ldots A^{n-1}B]) < n,$$

we can change coordinates as is shown in EE363 Lecture 6,

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = T^{-1}B = \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix}.$$
Let $\lambda$ be an eigenvalue of $\tilde{A}_{22}$, and $w_{22}$ be an associated left eigenvector. Define

$$w = T^{-T} \begin{bmatrix} 0 \\ w_{22} \end{bmatrix} \neq 0,$$

and so

$$w^T A = \begin{bmatrix} 0 \\ w_{22} \end{bmatrix}^T T^{-1} \left( T \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} T^{-1} \right)$$

$$= \begin{bmatrix} 0 & w_{22} \tilde{A}_{22} \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} 0 & \lambda w_{22} \end{bmatrix} T^{-1}$$

$$= \lambda \begin{bmatrix} 0 & w_{22} \end{bmatrix} T^{-1}$$

$$= \lambda w^T$$

similarly,

$$w^T B = \begin{bmatrix} 0 \\ w_{22} \end{bmatrix}^T T^{-1} \left( T \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & w_{22} \end{bmatrix} \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix}$$

$$= 0. \quad \Box$$

**Example:** Using $(A, B)$ from the previous example, construct a left eigenvalue of $A$ that is orthogonal to the columns of $B$

**Solution:** First, we form the controllability matrix

$$C = \begin{bmatrix} B & AB & A^2 B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

from where we can construct $T$,

$$T = \begin{bmatrix} 1 & -3 & 1 \\ -3 & 7 & 0 \\ -7 & -15 & 0 \end{bmatrix},$$

where we have concatenated $e_1$ to the last two columns of $C$. $\lambda = \tilde{A}_{22} = -3$, and choosing $w_{22} = 1$, we get

$$w = T^{-T} \begin{bmatrix} 0 \\ 0 \\ w_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix}.$$ 

Check:

$$w^T A = \begin{bmatrix} 1 & 1.5 & 0.5 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix} = \lambda w^T,$$
Thus, we have found a left eigenvector of $A$, orthogonal to the columns of $B$.

**PBH Observability Test:** PBH observability test says $(C, A)$ is observable if and only if there exists no right eigenvector of $A$ orthogonal to the columns of $C$.

**Proof:** similar to proof of PBH observability criterion.

**Example:** Given $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$, find necessary and sufficient conditions for $(C, A)$ to be observable.

**Solution:** Suppose $v \in \mathbb{C}^n$ is an eigenvector of $A$. Using the PBH observability criterion, $(C, A)$ is observable if and only if
\[
Cv \neq 0,
\]
i.e., every eigenspace of $A$ does not intersect with the $\mathcal{N}(C)$, except at the origin. Since $A$ is diagonal, this implies that the columns of $C$ corresponding to the same diagonal elements of $A$ should be linearly independent.