Review Session 5

• a partial summary of the course

• no guarantees everything on the exam is covered here

• not designed to stand alone; use with the class notes
LQR

• balance good control and small input effort

• quadratic cost function

\[ J(U) = \sum_{\tau=0}^{N-1} \left( x_{\tau}^T Q x_{\tau} + u_{\tau}^T R u_{\tau} \right) + x_N^T Q_f x_N \]

• \( Q, Q_f \) and \( R \) are state cost, final state cost, input cost matrices
Solving LQR problems

• can solve as least-squares problem

• solve more efficiently with dynamic programming: use value function

\[
V_t(z) = \min_{u_t, \ldots, u_{N-1}} \sum_{\tau=t}^{N-1} \left( x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + x_N^T Q_f x_N
\]

subject to \( x_t = z, x_{\tau+1} = A x_\tau + B u_\tau, \tau = t, \ldots, T \)

• \( V_t(z) \) is the minimum LQR cost-to-go from state \( z \) at time \( t \)

• can show by recursion that \( V_t(z) = z^T P_t z; u_{t, \text{lqr}} = K_t x_t \)

• get Riccati recursion, runs backwards in time
Steady-state LQR

• usually $P_t$ in value function converges rapidly as $t$ decreases below $N$

• steady-state value $P_{ss}$ satisfies

$$P_{ss} = Q + A^T P_{ss} A - A^T P_{ss} B (R + B^T P_{ss} B)^{-1} B^T P_{ss} A$$

• this is the discrete-time algebraic Riccati equation (ARE)

• for $t$ not close to horizon $N$, LQR optimal input is approximately a linear, constant state feedback
LQR extensions

- time-varying systems
- time-varying cost matrices
- tracking problems (with state/input offsets)
- Gauss-Newton LQR for nonlinear dynamical systems
- can view LQR as solution of constrained minimization problem, via Lagrange multipliers
Infinite horizon LQR

• problem becomes: choose $u_0, u_1, \ldots$ to minimize

$$J = \sum_{\tau=0}^{\infty} (x_\tau^T Q x_\tau + u_\tau^T R u_\tau)$$

• infinite dimensional problem

• possibly no solution in general

• if $(A, B)$ is controllable, then for any $x^{\text{init}}$, there’s a length-$n$ input sequence that steers $x$ to zero and keeps it there
Hamilton-Jacobi equation

• define value function \( V(z) = z^T P z \) as minimum LQR cost-to-go

• satisfies Hamilton-Jacobi equation

\[
V(z) = \min_w (z^T Q z + w^T R w + V(A z + B w)) ,
\]

• after minimizing over \( w \), HJ equation becomes

\[
z^T P z = z^T Q z + w^* T R w^* + (A z + B w^*)^T P (A z + B w^*)
\]

\[
= z^T (Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A) z
\]

• holds for all \( z \), so \( P \) satisfies the ARE (thus, constant state feedback)

\[
P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A
\]
Receding-horizon LQR control

- find sequence that minimizes first $T$-step-ahead LQR cost from current position then use just the first input

- in general, optimal $T$-step-ahead LQR control has constant state feedback

- state feedback gain converges to infinite horizon optimal as horizon becomes long (assuming controllability)

- closed loop system is stable if $(Q, A)$ observable and $(A, B)$ controllable
Continuous-time LQR

• choose \( u : [0, T] \rightarrow \mathbb{R}^m \) to minimize

\[
J = \int_0^T \left( x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) d\tau + x(T)^T Q_f x(T)
\]

• infinite dimensional problem

• can solve via dynamic programming, \( V_t \) again quadratic; \( P_t \) found from a differential equation, running backwards in time

• LQR optimal \( u \) easily expressed in terms of \( P_t \)

• can also handle time-varying/tracking problems
Continuous-time LQR in steady-state

- usually $P_t$ converges rapidly as $t$ decreases below $T$

- limit $P_{ss}$ satisfies continuous-time ARE

$$A^T P + PA - PBR^{-1}B^TP + Q = 0$$

- can solve using Riccati differential equation, or directly, via Hamiltonian

- for $t$ not near $T$, LQR optimal input is approximately a linear constant state feedback

- (can also derive via discretization or Lagrange multipliers)
Linear quadratic stochastic control

- add IID process noise $w_t$: $x_{t+1} = Ax_t + Bu_t + w_t$

- objective becomes

$$J = \mathbb{E}\left( \sum_{t=0}^{N-1} (x_t^T Q x_t + u_t^T R u_t) + x_N^T Q f x_N \right)$$

- choose input to minimize $J$, after knowing the current state, but before knowing the disturbance

- can solve via dynamic programming

- optimal policy is linear state feedback (same form as deterministic LQR)

- strangely, optimal policy is the same as LQR, doesn’t depend on $X$, $W$
Invariant subspaces

• $\mathcal{V}$ is $A$-invariant if $A\mathcal{V} \subseteq \mathcal{V}$, i.e., $v \in \mathcal{V} \implies Av \in \mathcal{V}$

• e.g., controllable/unobservable subspaces for linear systems

• if $\mathcal{R}(M)$ is $A$-invariant, then there is a matrix $X$ such that $AM = MX$

• converse is also true: if there is an $X$ such that $AM = MX$, then $\mathcal{R}(M)$ is $A$-invariant
PBH controllability criterion

- \((A, B)\) is controllable if and only if

\[
\text{Rank} \begin{bmatrix} sI - A & B \end{bmatrix} = n \quad \text{for all} \quad s \in \mathbb{C}
\]

or,

- \((A, B)\) is uncontrollable if and only if there is a \(w \neq 0\) with

\[
w^T A = \lambda w^T, \quad w^T B = 0
\]

i.e., a left eigenvector is orthogonal to columns of \(B\)

- mode associated with left eigenvector \(w\) is uncontrollable if \(w^T B = 0\),
PBH observability criterion

- \((C, A)\) is observable if and only if

\[
\text{Rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \text{ for all } s \in \mathbb{C}
\]

or,

- \((C, A)\) is unobservable if and only if there is a \(v \neq 0\) with

\[
Av = \lambda v, \quad Cv = 0
\]

i.e., a (right) eigenvector is in the nullspace of \(C\)

- mode associated with right eigenvector \(v\) is unobservable if \(Cv = 0\)
Estimation

• minimum mean-square estimator (MMSE) is, in general, $E(x|y)$

• for jointly Gaussian $x$ and $y$, MMSE estimator of $x$ is affine function of $y$

$$\hat{x} = \phi_{\text{mmse}}(y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

• when $x, y$ aren’t jointly Gaussian, best linear unbiased estimator is

$$\hat{x} = \phi_{\text{blu}}(y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

• $\phi_{\text{blu}}$ is unbiased ($E\hat{x} = E\bar{x}$), often works well, has MMSE among all affine estimators

• given $A, \Sigma_x, \Sigma_v$, can evaluate $\Sigma_{\text{est}}$ before knowing measurements (can do experiment design)
Linear system with stochastic process

- covariance $\Sigma_x(t)$ satisfies a Lyapunov-like linear dynamical system

$$\Sigma_x(t + 1) = A\Sigma_x(t)A^T + B\Sigma_u(t)B^T + A\Sigma_{xu}(t)B^T + B\Sigma_{ux}(t)A^T$$

- if $\Sigma_{xu}(t) = 0$ ($x$ and $u$ uncorrelated), we have the Lyapunov iteration

$$\Sigma_x(t + 1) = A\Sigma_x(t)A^T + B\Sigma_u(t)B^T$$

- if (and only if) $A$ is stable, converges to steady-state covariance which satisfies the Lyapunov equation

$$\Sigma_x = A\Sigma_x A^T + B\Sigma_u B^T$$
Kalman filter

- estimate current or next state, based on current and past outputs
- recursive, so computationally efficient (can express as Riccati recursion)
- measurement update
  \[
  \hat{x}_{t|t} = \hat{x}_{t|t-1} + \Sigma_{t|t-1}C^T \left( C\Sigma_{t|t-1}C^T + V \right)^{-1} (y_t - C\hat{x}_{t|t-1})
  \]
  \[
  \Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C^T \left( C\Sigma_{t|t-1}C^T + V \right)^{-1} C\Sigma_{t|t-1}
  \]
- time update
  \[
  \hat{x}_{t+1|t} = A\hat{x}_{t|t}, \quad \Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W
  \]
- can compute \( \Sigma_{t|t-1} \) before any observations are made
- steady-state error covariance satisfies ARE
  \[
  \hat{\Sigma} = A\hat{\Sigma}A^T + W - A\hat{\Sigma}C^T \left( C\hat{\Sigma}C^T + V \right)^{-1} C\hat{\Sigma}A^T
  \]
Approximate nonlinear filtering

• in general, exact solution is impractical; requires propagating infinite dimensional conditional densities

• extended Kalman filter: use affine approximations of nonlinearities, Gaussian model

• other methods (e.g., particle filters): based on Monte Carlo methods that sample the random variables

• usually heuristic, unless problems are very small
**Conservation and dissipation**

- A set $C \subseteq \mathbb{R}^n$ is invariant with respect to autonomous, time-invariant, nonlinear $\dot{x} = f(x)$ if for every trajectory $x$,

$$x(t) \in C \implies x(\tau) \in C \text{ for all } \tau \geq t$$

- Every trajectory that enters or starts in $C$ must stay there.

- Scalar valued function $\phi$ is a conserved quantity for $\dot{x} = f(x)$ if for every trajectory $x$, $\phi(x(t))$ is constant.

- $\phi$ is a dissipated quantity for $\dot{x} = f(x)$ if for every trajectory $x$, $\phi(x(t))$ is (weakly) decreasing.
Quadratic functions and linear dynamical systems

continuous time: linear system \( \dot{x} = Ax \), quadratic form \( \phi(z) = z^T P z \)

- \( \phi \) is conserved if and only if \( A^T P + PA = 0 \)
- \( \phi \) is dissipated if and only if \( A^T P + PA \leq 0 \)

discrete time: linear system \( x_{t+1} = Ax_t \), quadratic form \( \phi(z) = z^T P z \)

- \( \phi \) is conserved if and only if \( A^T PA - P = 0 \)
- \( \phi \) is dissipated if and only if \( A^T PA - P \leq 0 \)
Stability

consider nonlinear time-invariant system \( \dot{x} = f(x) \)

- \( x_e \) is an equilibrium point if \( f(x_e) = 0 \)

- system is globally asymptotically stable (GAS) if for every trajectory \( x \), \( x(t) \to x_e \) as \( t \to \infty \)

- system is locally asymptotically stable (LAS) near or at \( x_e \), if there is an \( R > 0 \) such that \( \|x(0) - x_e\| \leq R \implies x(t) \to x_e \) as \( t \to \infty \)

- for linear systems (with \( x_e = 0 \)), LAS \( \iff \) GAS \( \iff \Re \lambda_i(A) < 0 \)
Energy and dissipation functions

consider nonlinear time-invariant system $\dot{x} = f(x)$, function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

• define $\dot{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ as $\dot{V}(z) = \nabla V(z)^T f(z)$

• $\dot{V}(z)$ gives $\frac{d}{dt} V(x(t))$ when $z = x(t)$, $\dot{x} = f(x)$

• can think of $V$ as generalized energy function, $-\dot{V}$ as the associated generalized dissipation function

• $V$ is positive definite if $V(z) \geq 0$ for all $z$, $V(z) = 0$ if and only if $z = 0$ and all sublevel sets of $V$ are bounded ($V(z) \rightarrow \infty$ as $z \rightarrow \infty$)
Lyapunov theory

• used to make conclusions about of system trajectories, without finding the trajectories

• boundedness: if there is a (Lyapunov function) $V$ with all sublevel sets bounded, and $\dot{V}(z) \leq 0$ for all $z$, then all trajectories are bounded

• global asymptotic stability: if there is a positive definite $V$ with $\dot{V}(z) < 0$ for all $z \neq 0$ and $\dot{V}(0) = 0$, then every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$

• exponential stability: if there is a positive definite $V$, and constant $\alpha > 0$ with $\dot{V}(z) \leq -\alpha V(z)$ for all $z$, then there is an $M$ such that every trajectory satisfies $\|x(t)\| \leq M e^{-\alpha t/2} \|x(0)\|$
Lasalle’s theorem

- can conclude GAS of a system with only $\dot{V} \leq 0$ and an observability-type condition

- if there is a positive definite $V$ with $\dot{V}(z) \leq 0$, and the only solution of $\dot{w} = f(w)$, $V(w) = 0$ is $w(t) = 0$ for all $t$, then the system is GAS

- requires time-invariance
Converse Lyapunov theorems

- if a linear system is GAS, there is a quadratic Lyapunov function that proves it

- if a system is globally exponentially stable, there is a Lyapunov function that proves it
Linear quadratic Lyapunov theory

- Lyapunov equation: $A^T P + PA + Q = 0$

- for linear system $\dot{x} = Ax$, if $V(z) = z^T P z$, then $\dot{V}(z) = (Az)^T P z + z^T P (Az) = -z^T Q z$

- if $z^T P z$ is the generalized energy, then $z^T Q z$ is the associated generalized dissipation

- boundedness: if $P > 0$, $Q \geq 0$, then all trajectories are bounded, and the ellipsoids $\{z \mid z^T P z \leq a\}$ are invariant

- stability: if $P > 0$, $Q > 0$, then the system is GAS

- an extension from Lasalle’s theorem: if $P > 0$, $Q \geq 0$ and $(Q, A)$ observable, then the system is GAS

- if $Q \geq 0$ and $P \not\geq 0$, then $A$ is not stable
The Lyapunov operator

- the Lyapunov operator is given by

\[ \mathcal{L}(P) = A^T P + PA \]

- if \( A \) is stable, Lyapunov operator is nonsingular

- if \( A \) has imaginary eigenvalue, then Lyapunov operator is singular

- thus, if \( A \) is stable, for any \( Q \) there is exactly one solution \( P \) of the Lyapunov equation \( A^T P + PA + Q = 0 \)

- efficient ways to solve the Lyapunov equation (review session 3)
The Lyapunov integral

• if $A$ is stable, explicit formula for solution of Lyapunov equation:

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \, dt$$

• if $A$ is stable, $P$ is unique solution of Lyapunov equation, then

$$V(z) = z^T P z = \int_0^\infty x(t)^T Q x(t) \, dt$$

(where $\dot{x} = Ax$ and $x(0) = z$)

• thus, $V(z)$ is cost-to-go from point $z$, and integral quadratic cost function with matrix $Q$

• can use to evaluate quadratic integrals
Further Lyapunov results

- All linear quadratic Lyapunov results have discrete-time counterparts.

- Discrete-time Lyapunov equation is

  \[ A^T P A - P + Q = 0 \]

  (if \( V(z) = z^T P z \), then \( \delta V(z) = -z^T Q z \))

- For a nonlinear system \( \dot{x} = f(x) \) with \( x_e \) an equilibrium point, if the linearized system near \( x_e \) is stable, then the nonlinear system is locally asymptotically stable (and nearly vice versa).
LMIs

- the Lyapunov inequality $A^TP + PA + Q \leq 0$ is an LMI in variable $P$

- $P$ satisfies the Lyapunov LMI if and only if the quadratic form
  \[ V(\dot{z}) = \dot{z}^TP\dot{z} \text{ satisfies } \dot{V}(z) \leq -z^TQz \]

- bounded-real LMI: if $P$ satisfies
  \[
  \begin{bmatrix}
  A^TP + PA + C^TC & PB \\
  B^TP & -\gamma^2I
  \end{bmatrix} \leq 0, \quad P \succeq 0
  \]

then the quadratic Lyapunov function $V(z) = z^TPz$ proves the RMS gain of the system is no more than $\gamma$
Using LMIs

• practical approach to strict matrix inequalities: if inequalities are homogeneous in $x$, replace $F_{\text{strict}}(x) > 0$ with $F_{\text{strict}}(x) \geq I$

• if inequalities aren’t homogeneous, replace $F_{\text{strict}}(x) > 0$ with $F_{\text{strict}}(x) \geq \epsilon I$, with $\epsilon$ small and positive

• if we have $\dot{x}(t) = A(t)x(t)$, with $A(t) \in \{A_1, \ldots, A_K\}$, can use multiple simultaneous LMIs to find a simultaneous Lyapunov function that establishes a property for all trajectories

• can’t be done analytically, but possible to do numerically

• more generally, can globally and efficiently solve SDPs:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0 \\
& \quad Ax = b
\end{align*}
\]
S-procedure

• for two quadratic forms, if and (with a constraint qualification) only if there is a $\tau \geq 0$ with $F_0 \geq \tau F_1$, then $z^T F_1 z \geq 0 \implies z^T F_0 z \geq 0$

• can also replace $\geq$ with $>$

• for multiple quadratic forms, if there are $\tau_1, \ldots, \tau_k \geq 0$ with

$$ F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k $$

then, for all $z$,

$$ z^T F_1 z \geq 0, \ldots, z^T F_k z \geq 0 \implies z^T F_0 z \geq 0 $$

• can solve using LMIs
Systems with sector nonlinearities

• a function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be in sector $[l, u]$ if for all $q \in \mathbb{R}$, $p = \phi(q)$ lies between $lq$ and $uq$

• a (single nonlinearity) Lur’e system has the form

$$\dot{x} = Ax + Bp, \quad q = Cx, \quad p = \phi(t, q)$$

where $\phi(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is in sector $[l, u]$ for each $t$

• goal: prove stability or bound using only the sector information
GAS of Lur’e system

• can express GAS of Lur’e system using quadratic Lyapunov function

\[ V(z) = z^T P z \]

as requiring \[ \dot{V} + \alpha V \leq 0 \], equivalent to

\[
\begin{bmatrix}
z \\
p
\end{bmatrix}^T
\begin{bmatrix}
A^T P + PA + \alpha P & PB \\
B^T P & 0
\end{bmatrix}
\begin{bmatrix}
z \\
p
\end{bmatrix} \leq 0
\]

whenever

\[
\begin{bmatrix}
z \\
p
\end{bmatrix}^T
\begin{bmatrix}
\sigma C^T C & -\nu C^T \\
-\nu C & 1
\end{bmatrix}
\begin{bmatrix}
z \\
p
\end{bmatrix} \leq 0
\]

• can convert this to the LMI (with variables \( P \) and \( \tau \))

\[
\begin{bmatrix}
A^T P + PA + \alpha P - \tau \sigma C^T C & PB + \tau \nu C^T \\
B^T P + \tau \nu C & -\tau
\end{bmatrix} \leq 0, \quad P \geq I
\]

• can sometimes extend to case with multiple nonlinearities
Perron-Frobenius theory

• a nonnegative matrix $A$ is regular if for some $k \geq 1$, $A^k > 0$
  (path of length $k$ from every node to every other node)

• if $A$ is regular, then there is a real, positive, strictly dominant, simple
  Perron-Frobenius eigenvalue $\lambda_{pf}$, with positive left and right
  eigenvectors

• if we only have $A \geq 0$, then there is an eigenvalue $\lambda_{pf}$ of $A$ that is real,
  nonnegative and (non-strictly) dominant, and has (possibly not unique)
  nonnegative left and right eigenvectors

• For a Markov chain with transition matrix $P$, if $P$ is regular, the
  distribution always converges to the unique invariant distribution $\pi > 0$,
  associated with a simple, dominant eigenvalue of 1

• rate of convergence depends on second largest eigenvalue magnitude
Max-min/min-max ratio characterization

- Perron-Frobenius eigenvalue is optimal value of two optimization problems

\[
\begin{align*}
\text{maximize} & \quad \min_i \frac{(Ax)_i}{x_i} \\
\text{subject to} & \quad x > 0
\end{align*}
\]

and

\[
\begin{align*}
\text{minimize} & \quad \max_i \frac{(Ax)_i}{x_i} \\
\text{subject to} & \quad x > 0
\end{align*}
\]

- the optimal \( x \) is the Perron-Frobenius eigenvector
Linear Lyapunov functions

• suppose $c > 0$, and consider the linear Lyapunov function $V(z) = c^T z$

• if $V(Az) \leq \delta V(z)$ for some $\delta < 1$ and all $z \geq 0$, then $V$ proves (nonnegative) trajectories converge to zero

• a nonnegative regular system is stable if and only if there is a linear Lyapunov function that proves it
Continuous time results

- $\mathbb{R}_+^n$ is invariant under $\dot{x} = Ax$ if and only if $A_{ij} \geq 0$ for $i \neq j$

- such matrices are called Metzler matrices

- $A$ has a real, dominant eigenvalue $\lambda_{\text{metzler}}$ that is real and has associated nonnegative left and right eigenvectors

- analogs exist with other discrete-time results
Exam advice

- five questions
- determine the topic(s) each question covers
- guess the form the problem statement should take
- manipulate (‘hammer’) the question into that standard form
- explain things as simply as possible; if your solution is extremely complicated, you’re probably missing something
- we’re not especially concerned about boundary conditions or edge cases, but mention any assumptions you make