Convex optimization examples

- multi-period processor speed scheduling
- minimum time optimal control
- grasp force optimization
- optimal broadcast transmitter power allocation
- phased-array antenna beamforming
- optimal receiver location
Multi-period processor speed scheduling

- processor adjusts its speed $s_t \in [s_{\text{min}}, s_{\text{max}}]$ in each of $T$ time periods
- energy consumed in period $t$ is $\phi(s_t)$; total energy is $E = \sum_{t=1}^{T} \phi(s_t)$
- $n$ jobs
  - job $i$ available at $t = A_i$; must finish by deadline $t = D_i$
  - job $i$ requires total work $W_i \geq 0$
- $\theta_{ti} \geq 0$ is fraction of processor effort allocated to job $i$ in period $t$
  
  \[
  \mathbf{1}^T \theta_t = 1, \quad \sum_{t=A_i}^{D_i} \theta_{ti}s_t \geq W_i
  \]
- choose speeds $s_t$ and allocations $\theta_{ti}$ to minimize total energy $E$
Minimum energy processor speed scheduling

- work with variables $S_{ti} = \theta_{ti} s_t$

$$s_t = \sum_{i=1}^{n} S_{ti}, \quad \sum_{t=A_i}^{D_i} S_{ti} \geq W_i$$

- solve convex problem

\[
\begin{align*}
\text{minimize} & \quad E = \sum_{t=1}^{T} \phi(s_t) \\
\text{subject to} & \quad s^\text{min} \leq s_t \leq s^\text{max}, \quad t = 1, \ldots, T \\
& \quad s_t = \sum_{i=1}^{n} S_{ti}, \quad t = 1, \ldots, T \\
& \quad \sum_{t=A_i}^{D_i} S_{ti} \geq W_i, \quad i = 1, \ldots, n
\end{align*}
\]

- a convex problem when $\phi$ is convex

- can recover $\theta^*_t$ as $\theta^*_{ti} = (1/s^*_t)S^*_t$
Example

- $T = 16$ periods, $n = 12$ jobs
- $s^{\text{min}} = 1$, $s^{\text{max}} = 6$, $\phi(s_t) = s_t^2$
- jobs shown as bars over $[A_i, D_i]$ with area $\propto W_i$
Optimal and uniform schedules

- uniform schedule: \( S_{ti} = W_i / (D_i - A_i + 1) \); gives \( E_{\text{unif}} = 204.3 \)
- optimal schedule: \( S_{ti}^* \); gives \( E^* = 167.1 \)
Minimum-time optimal control

• linear dynamical system:

\[ x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, \ldots, K, \quad x_0 = x^{\text{init}} \]

• inputs constraints:

\[ u_{\text{min}} \leq u_t \leq u_{\text{max}}, \quad t = 0, 1, \ldots, K \]

• minimum time to reach state \( x_{\text{des}} \):

\[ f(u_0, \ldots, u_K) = \min \{ T \mid x_t = x_{\text{des}} \text{ for } T \leq t \leq K + 1 \} \]
state transfer time $f$ is quasiconvex function of $(u_0, \ldots, u_K)$:

$$f(u_0, u_1, \ldots, u_K) \leq T$$

if and only if for all $t = T, \ldots, K + 1$

$$x_t = A^t x_{\text{init}} + A^{t-1} Bu_0 + \cdots + Bu_{t-1} = x_{\text{des}}$$

*i.e.*, sublevel sets are affine

**minimum-time optimal control problem:**

minimize $f(u_0, u_1, \ldots, u_K)$

subject to $u_{\text{min}} \leq u_t \leq u_{\text{max}}, \quad t = 0, \ldots, K$

with variables $u_0, \ldots, u_K$

a quasiconvex problem; can be solved via bisection
Minimum-time control example

- force $(u_t)_1$ moves object modeled as 3 masses (2 vibration modes)
- force $(u_t)_2$ used for active vibration suppression
- goal: move object to commanded position as quickly as possible, with

$$|(u_t)_1| \leq 1, \quad |(u_t)_2| \leq 0.1, \quad t = 0, \ldots, K$$
Ignoring vibration modes

- treat object as single mass; apply only \( u_1 \)
- analytical ('bang-bang') solution
With vibration modes

- no analytical solution
- a quasiconvex problem; solved using bisection
Grasp force optimization

- choose $K$ grasping forces on object
  - resist external wrench
  - respect friction cone constraints
  - minimize maximum grasp force

- convex problem (second-order cone program):

  minimize $\max_i \| f^{(i)} \|_2$

  subject to
  \[
  \sum_i Q^{(i)} f^{(i)} = f^{\text{ext}} \\
  \sum_i p^{(i)} \times (Q^{(i)} f^{(i)}) = \tau^{\text{ext}} \\
  \mu_i f_3^{(i)} \geq \left( f_1^{(i)} + f_2^{(i)} \right)^{1/2}
  \]

  variables $f^{(i)} \in \mathbb{R}^3$, $i = 1, \ldots, K$ (contact forces)
Example
Optimal broadcast transmitter power allocation

- $m$ transmitters, $mn$ receivers all at same frequency
- transmitter $i$ wants to transmit to $n$ receivers labeled $(i, j)$, $j = 1, \ldots, n$
- $A_{ijk}$ is path gain from transmitter $k$ to receiver $(i, j)$
- $N_{ij}$ is (self) noise power of receiver $(i, j)$
- variables: transmitter powers $p_k$, $k = 1, \ldots, m$
at receiver \((i, j)\):

- signal power:
  \[ S_{ij} = A_{iij}p_i \]

- noise plus interference power:
  \[ I_{ij} = \sum_{k \neq i} A_{ijk}p_k + N_{ij} \]

- signal to interference/noise ratio (SINR): \( S_{ij}/I_{ij} \)

**Problem:** choose \( p_i \) to maximize smallest SINR:

\[
\begin{align*}
\text{maximize} & \quad \min_{i,j} \frac{A_{iij}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}} \\
\text{subject to} & \quad 0 \leq p_i \leq p_{\text{max}}
\end{align*}
\]

... a (generalized) linear fractional program
Phased-array antenna beamforming

- omnidirectional antenna elements at positions \((x_1, y_1), \ldots, (x_n, y_n)\)

- unit plane wave incident from angle \(\theta\) induces in \(i\)th element a signal 
  \[ e^{j(x_i \cos \theta + y_i \sin \theta - \omega t)} \]

  
  
  \((j = \sqrt{-1}, \text{ frequency } \omega, \text{ wavelength } 2\pi)\)
• demodulate to get output $e^{j(x_i \cos \theta + y_i \sin \theta)} \in \mathbb{C}$

• linearly combine with complex weights $w_i$:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos \theta + y_i \sin \theta)}$$

• $y(\theta)$ is (complex) antenna array gain pattern

• $|y(\theta)|$ gives sensitivity of array as function of incident angle $\theta$

• depends on design variables $\text{Re} \ w, \text{Im} \ w$
  (called antenna array weights or shading coefficients)

**design problem:** choose $w$ to achieve desired gain pattern
**Sidelobe level minimization**

make $|y(\theta)|$ small for $|\theta - \theta_{\text{tar}}| > \alpha$

($\theta_{\text{tar}}$: target direction; $2\alpha$: beamwidth)

via **least-squares** (discretize angles)

\[
\begin{align*}
\text{minimize} & \quad \sum_{i} |y(\theta_i)|^2 \\
\text{subject to} & \quad y(\theta_{\text{tar}}) = 1
\end{align*}
\]

(sum is over angles outside beam)

least-squares problem with two (real) linear equality constraints
\[ \theta_{\text{tar}} = 30^\circ \]

\[ |y(\theta)| \]

sidelobe level
**minimize sidelobe level**  (discretize angles)

\[
\begin{align*}
\text{minimize } & \quad \max_i |y(\theta_i)| \\
\text{subject to } & \quad y(\theta_{\text{tar}}) = 1
\end{align*}
\]

(max over angles outside beam)

Can be cast as SOCP

\[
\begin{align*}
\text{minimize } & \quad t \\
\text{subject to } & \quad |y(\theta_i)| \leq t \\
& \quad y(\theta_{\text{tar}}) = 1
\end{align*}
\]
\[
\theta_{\text{tar}} = 30^\circ
\]

\[
|y(\theta)|
\]

sidelobe level
Extensions

convex (& quasiconvex) extensions:

- \( y(\theta_0) = 0 \) (null in direction \( \theta_0 \))
- \( w \) is real (amplitude only shading)
- \( |w_i| \leq 1 \) (attenuation only shading)
- minimize \( \sigma^2 \sum_{i=1}^{n} |w_i|^2 \) (thermal noise power in \( y \))
- minimize beamwidth given a maximum sidelobe level

nonconvex extension:

- maximize number of zero weights
Optimal receiver location

• $N$ transmitter frequencies $1, \ldots, N$
• transmitters at locations $a_i, b_i \in \mathbb{R}^2$ use frequency $i$
• transmitters at $a_1, a_2, \ldots, a_N$ are the wanted ones
• transmitters at $b_1, b_2, \ldots, b_N$ are interfering
• receiver at position $x \in \mathbb{R}^2$
• (signal) receiver power from $a_i$: $\|x - a_i\|^{-\alpha}_2$ ($\alpha \approx 2.1$)

• (interfering) receiver power from $b_i$: $\|x - b_i\|^{-\alpha}_2$ ($\alpha \approx 2.1$)

• worst signal to interference ratio, over all frequencies, is

$$S/I = \min_i \frac{\|x - a_i\|^{-\alpha}_2}{\|x - b_i\|^{-\alpha}_2}$$

• what receiver location $x$ maximizes $S/I$?
S/I is quasiconcave on \( \{x \mid S/I \geq 1\} \), i.e., on

\[ \{x \mid \|x - a_i\|_2 \leq \|x - b_i\|_2, \ i = 1, \ldots, N\} \]

can use bisection; every iteration is a convex quadratic feasibility problem