EE364b Homework 1

Due Friday 4/12 at 11:59pm.

1.1 (a) For each of the following convex functions, determine the subdifferential set at the specified point.

i. $f(x_1, x_2, x_3) = \max\{|x_1|, |x_2|, |x_3|\}$ at $(x_1, x_2, x_3) = (0, 0, 0)$.

ii. $f(x) = e^{|x|}$ at $x = 0$ (x is a scalar).

iii. $f(x_1, x_2) = \max\{x_1 + x_2 - 1, x_1 - x_2 + 1\}$ at $(x_1, x_2) = (1, 1)$.

(b) For each of the following convex functions, explain how to calculate a subgradient at a given x.

i. $f(x) = \max_{i=1,\dots,m}(a_i^T x + b_i)$.

ii. $f(x) = \max_{i=1,\dots,m} |a_i^T x + b_i|$.

iii. $f(x) = \max_{i=1,\dots,m} \left(-\log\left(a_i^T x + b_i\right)\right)$. You may assume x is in the domain of f.

iv. $f(x) = \sup_{0 \le t \le 1} p(t)$, where $p(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$.

v. $f(x) = x_{[1]} + \cdots + x_{[k]}$, where $x_{[i]}$ denotes the ith largest element of the vector x.

vi. $f(x) = \inf_{Ay \preceq b} \|x - y\|^2$, *i.e.*, the square of the distance of x to the polyhedron defined by $Ay \preceq b$. You may assume that the inequalities $Ay \preceq b$ are strictly feasible.

vii. $f(x) = \sup_{Ay \preceq b} y^T x$, *i.e.*, the optimal value of an LP as a function of the cost vector. (You can assume that the polyhedron defined by $Ay \preceq b$ is bounded.)

1.2 *Convex functions that are not subdifferentiable.* Verify that the following functions, defined on the interval $[0, \infty)$, are convex, but not subdifferentiable at $x = 0$.

(a) $f(0) = 1$, and $f(x) = 0$ for $x > 0$.

(b) $f(x) = -\sqrt{x}$.

1.5 *Subgradient optimality conditions for nondifferentiable inequality constrained optimization.* Consider the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m, \end{array}$$

with variable $x \in \mathbf{R}^n$. We *do not* assume that $f_0, \dots, f_m$ are convex. Suppose that $\tilde{x}$ and $\tilde{\lambda} \succeq 0$ satisfy primal feasibility,

$$f_i(\tilde{x}) \le 0, \quad i = 1, \dots, m,$$

dual feasibility,

$$0 \in \partial f_0(\tilde{x}) + \sum_{i=1}^{m} \tilde{\lambda}_i \partial f_i(\tilde{x}),$$

and the complementarity condition

$$\tilde{\lambda}_i f_i(\tilde{x}) = 0, \quad i = 1, \dots, m.$$

Show that $\tilde{x}$ is optimal, using only a simple argument, and definition of subgradient. Recall that we do not assume the functions $f_0, \dots, f_m$ are convex.

1.9 *Conjugacy and subgradients.* In this question, we show how conjugate functions are related to subgradients. Let f be convex and recall that its conjugate is $f^*(v) = \sup_x \{v^T x - f(x)\}$. Prove the following:

(a) For any v we have $v^T x \leq f(x) + f^*(v)$ (this is sometimes called Young's inequality).

(b) We have $g^T x = f(x) + f^*(g)$ if and only if $g \in \partial f(x)$.

Note that (you do not need to prove this) if f is closed, so that $f(x) = f^{**}(x)$, result (b) implies the duality relationship that $g \in \partial f(x)$ if and only if $x \in \partial f^*(g)$ if and only if $g^T x = f(x) + f^*(g)$.

1.10 If a function has a unique subgradient at a given point, is the function differentiable at that point? Provide a proof or a counter example.

1.12 Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x_1, x_2) = \max\left\{\frac{1}{2}\|x\|^2 - x_1, \frac{1}{2}\|x\|^2 + x_1\right\}$$

(a) Determine the subdifferential set $\partial f(x)$ for $x \in \mathbf{R}^2$.

(b) Are the subgradients uniformly bounded over $x \in \mathbf{R}^2$? Would your answer change if x is restricted to lie in the set $X = \{x \in \mathbf{R}^2 \mid \|x\| \leq 1\}$? If yes, provide a bound for the subgradient norms.

2.3 *Matrix norm approximation.* We consider the problem of approximating a given matrix $B \in \mathbf{R}^{p \times q}$ as a linear combination of some other given matrices $A_i \in \mathbf{R}^{p \times q}$, $i = 1, \dots, n$, as measured by the matrix norm (maximum singular value):

$$\text{minimize} \quad \|x_1 A_1 + \cdots + x_n A_n - B\|.$$

(a) Explain how to find a subgradient of the objective function at x.

(b) Generate a random instance of the problem with $n = 5$, $p = 3$, $q = 6$. Use CVX to find the optimal value $f^\star$ of the problem. Use a subgradient method to solve the problem, starting from $x = 0$. Plot $f - f^\star$ versus iteration. Experiment with several step size sequences.