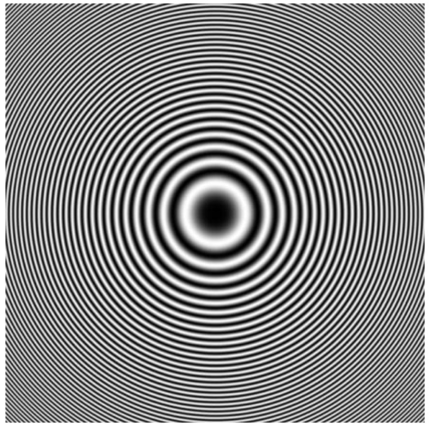
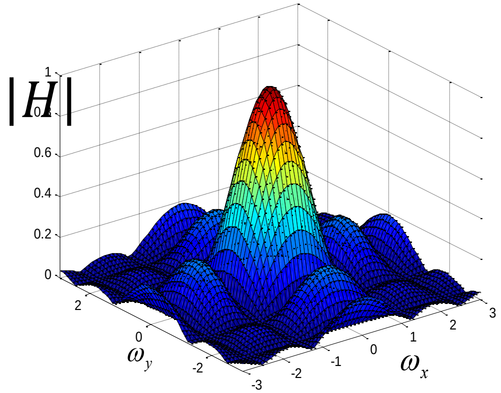
# Linear Image Processing and Filtering

- Math of 2-d linear systems
- Separability
- Shift-invariant linear systems, 2-d convolution
- Integral image for fast box-filtering
- 2-d frequency response
- Blurring and sharpening
- Zoneplate
- Image sampling/aliasing
- Wiener filtering
- Nonlinear noise reduction/sharpening





# Linear image processing

- Image processing system $S(.)$ is linear, iff superposition principle holds:

$$S\left(\alpha \cdot f[x,y] + \beta \cdot g[x,y]\right) = \alpha \cdot S\left(f[x,y]\right) + \beta \cdot S\left(g[x,y]\right) \quad \text{for all } \alpha, \beta \in \mathrm{R}$$

- Any linear image processing system can be written as

$$\vec{g} = H\vec{f}$$

<u>Note:</u> matrix $H$ need not be square.

by sorting pixels into a column vector

$$\vec{f} = \begin{pmatrix} f[0,0] & f[1,0] & \cdots & f[N-1,0] & f[0,1] & \cdots & f[N-1,1] & \cdots & \cdots & f[0,L-1] & \cdots & f[N-1,L-1] \end{pmatrix}^T$$

# Impulse response

- Another way to represent any linear image processing scheme

$$\boxed{g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h[x,\alpha,y,\beta]}$$

impulse response

- Input: unit impulse at pixel $[a,b]$

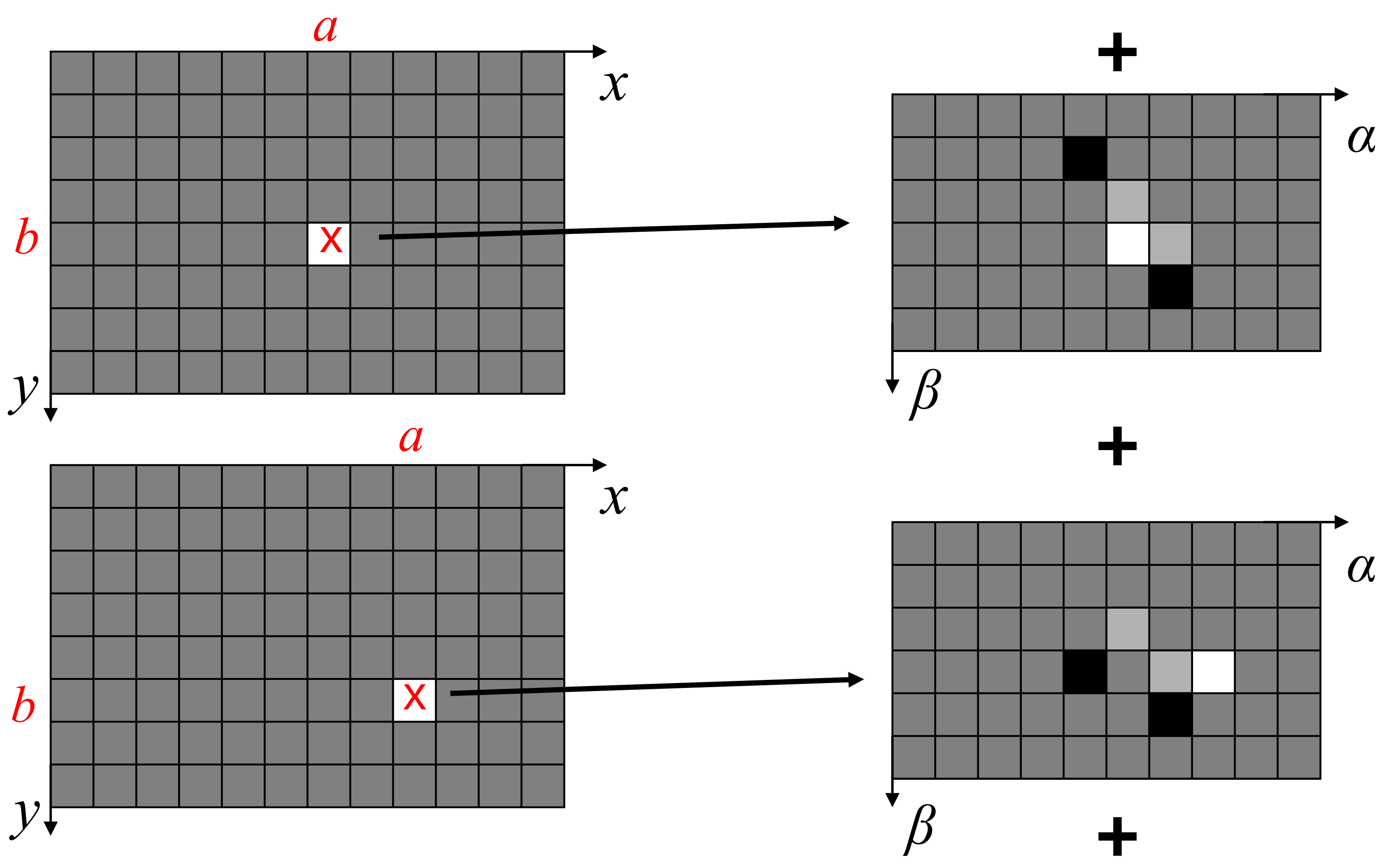
$$f[x,y]=\delta[x-a,y-b]=\begin{cases}1 & x=a \wedge y=b\\ 0 & else\end{cases}$$

- Output: impulse response

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1}\delta[x-a,y-b]\cdot h[x,\alpha,y,\beta]=h[a,\alpha,b,\beta]$$

# Interpretation #1: superposition of impulse responses

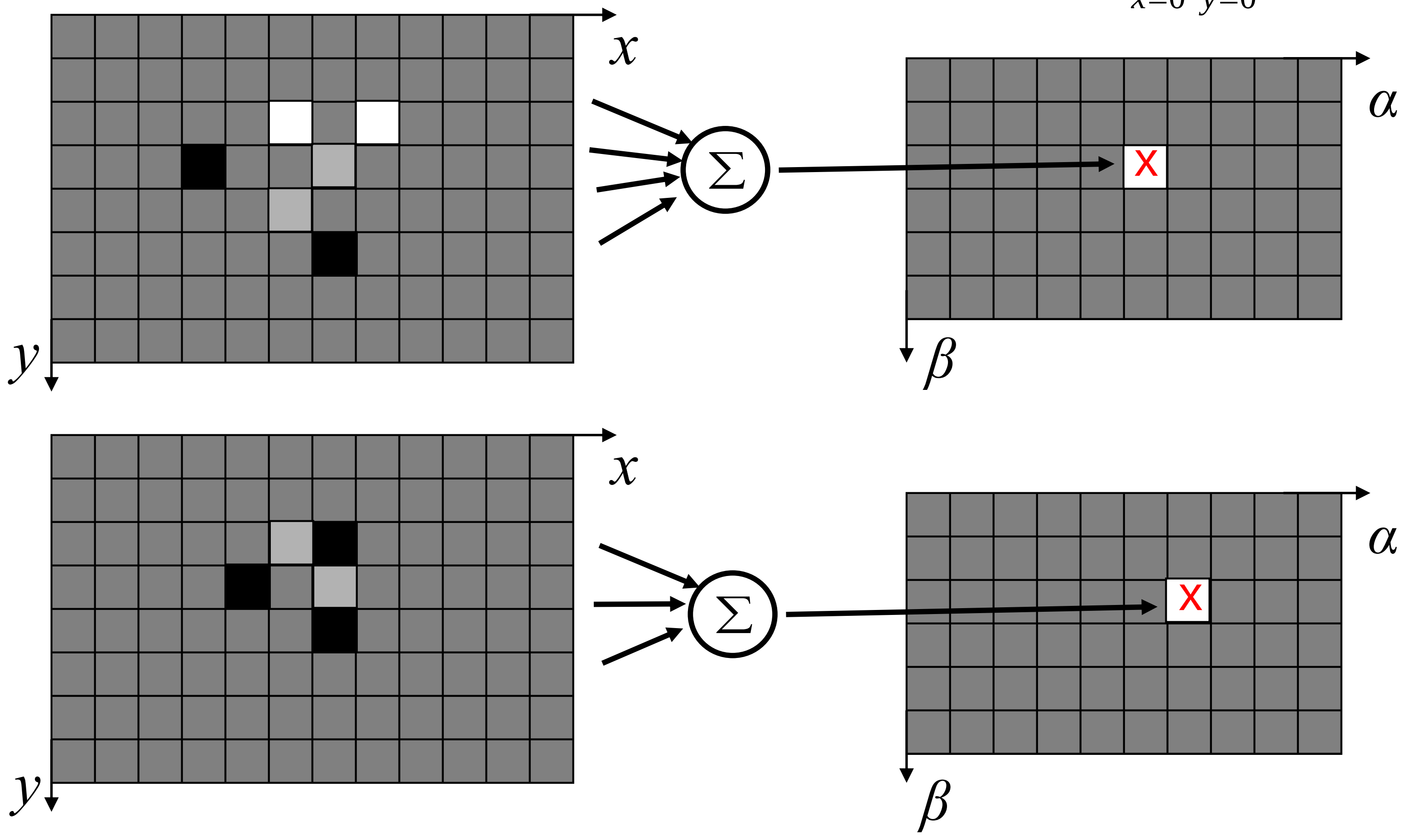
$\delta[x-a,y-b]f[a,b]$ $\qquad$ $g[\alpha,\beta]=h[a,\alpha,b,\beta]\cdot f[a,b]$



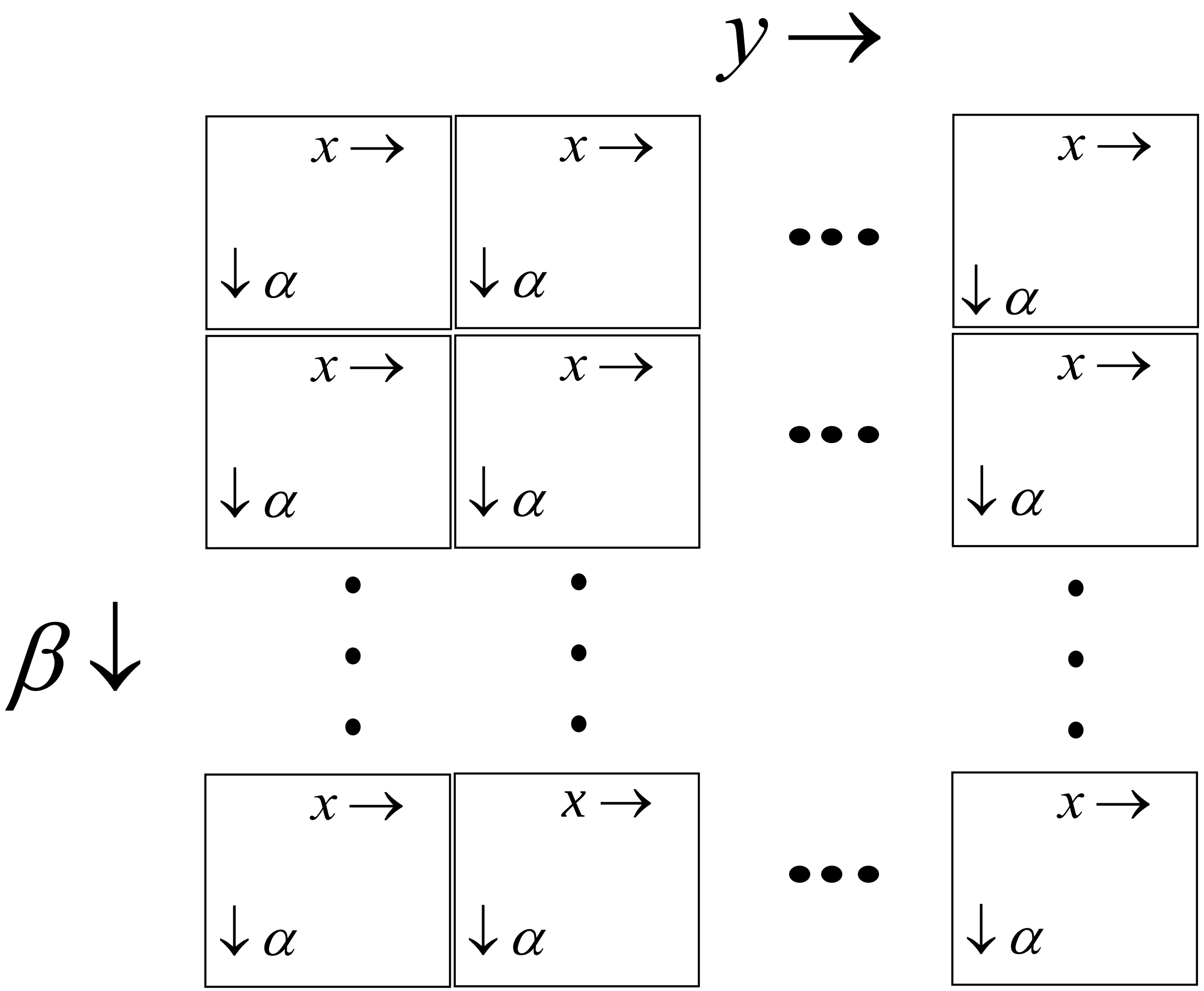
# Interpretation #2: linear combination of input values

$$f[x,y]\cdot h[x,\alpha,y,\beta]$$

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h[x,\alpha,y,\beta]$$



# Relationship of $H$ and $h[x,\alpha,y,\beta\ ]$



# Separable linear image processing

- Impulse response is separable in $[x,\alpha]$ and $[y,\beta]$, i.e., can be written as

$$g[\alpha,\beta]=\sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_x[x,\alpha]h_y[y,\beta]$$

- Processing can be carried out
  - row by row, then column by column

$$g[\alpha,\beta]=\sum_{y=0}^{L-1} h_y[y,\beta]\sum_{x=0}^{N-1} f[x,y]\cdot h_x[x,\alpha]$$

  - column by column, then row by row

$$g[\alpha,\beta]=\sum_{x=0}^{N-1} h_x[x,\alpha]\sum_{y=0}^{L-1} f[x,y]\cdot h_y[y,\beta]$$

# Separable linear image processing (cont.)

- If the digital input and output images are written as a matrices $\mathbf{f}$ and $\mathbf{g}$, we can conveniently write

$$\mathbf{g} = \mathbf{H_y}^{\mathbf{T}} \cdot \mathbf{f} \cdot \mathbf{H_x}$$

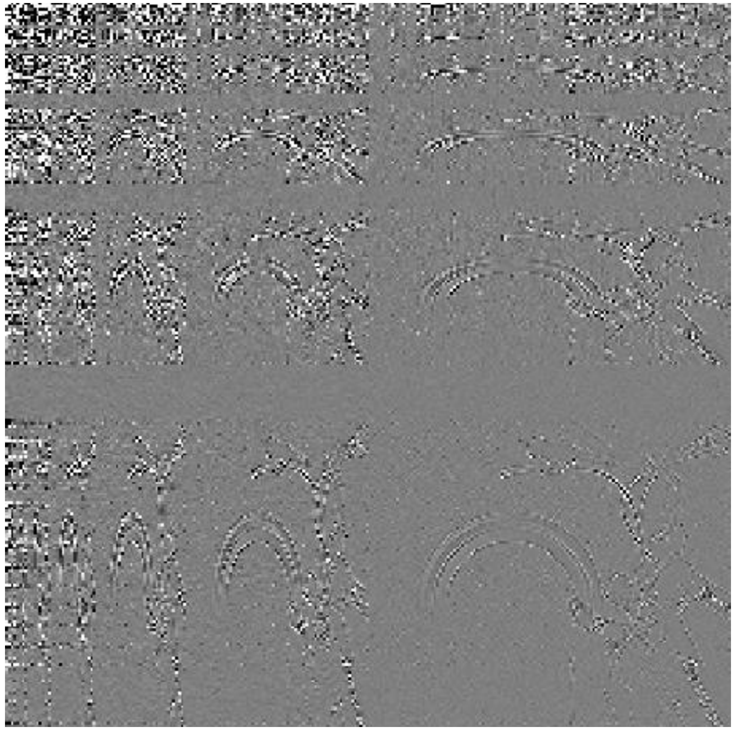
$$\mathbf{H}_y = \begin{bmatrix} h_y[0,0] & h_y[0,1] & \cdots & h_y[0,L_g-1] \\ h_y[1,0] & h_y[1,1] & \cdots & h_y[1,L_g-1] \\ \vdots & \vdots & & \vdots \\ h_y[L-1,0] & h_y[L-1,1] & \cdots & h_y[L-1,L_g-1] \end{bmatrix}$$

$$\mathbf{H}_x = \begin{bmatrix} h_x[0,0] & h_x[0,1] & \cdots & h_x[0,N_g-1] \\ h_x[1,0] & h_x[1,1] & \cdots & h_x[1,N_g-1] \\ \vdots & \vdots & & \vdots \\ h_x[N-1,0] & h_x[N-1,1] & \cdots & h_x[N-1,N_g-1] \end{bmatrix}$$

- Output image $\mathbf{g}$ has size $L_g$ x $N_g$
- If the operator does not change image size, $\mathbf{H_x}$ and $\mathbf{H_y}$ are square matrices

# Example: Separable Haar transform

Original *Bike*
*256x256*



*256x256*
Haar transform
$H_x = H_y = Hr_{256}$

# Haar transform

- Haar transform matrix for sizes $N=2,4,8$

$$\mathbf{Hr}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{Hr}_4 = \frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

$$\mathbf{Hr}_8 = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{pmatrix}$$

- Can be computed by taking sums and differences
- Fast algorithms by recursively applying $\mathbf{Hr}_2$

# Example: Subsampling

- Image subsampling 2:1 horizontally and vertically
- Small input image of size 8x8, output image size 4x4

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Example: Subsampling (cont.)

- A somewhat better technique for 2:1 image size reduction

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

# Side by side comparison

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

# Another example: filtering

- Each pixel is replaced by the average of two horizontally (vertically) neighboring pixels

$$\mathbf{H}_x = \mathbf{H}_y = \begin{bmatrix} 0.5 & 0 & & \cdots & & & & 0 \\ 0.5 & 0.5 & & & & & & \\ 0 & 0.5 & 0.5 & & & & & \\ & & 0.5 & 0.5 & \ddots & & & \vdots \\ & & & 0.5 & 0.5 & & & \\ \vdots & & & \ddots & 0.5 & 0.5 & & \\ & & & & & 0.5 & 0.5 & 0 \\ 0 & & & \cdots & & 0 & 0.5 & 0.5 \end{bmatrix}$$

- Shift-invariant operation (except for image boundary)

# Shift-invariant systems and Toeplitz matrices

- For a separable, shift-invariant, linear system

$$h_x[x,\alpha] = h_{siv/x}[\alpha - x] \quad \text{and} \quad h_y[y,\beta] = h_{siv/y}[\beta - y]$$

- Matrices $\mathbf{H_x}$ and $\mathbf{H_y}$ are square, and Toeplitz matrices, e.g.,

$$\mathbf{H_x} = \begin{bmatrix} h_{siv/x}[0] & h_{siv/x}[1] & \cdots & h_{siv/x}[N-1] \\ h_{siv/x}[-1] & h_{siv/x}[0] & \cdots & h_{siv/x}[N-2] \\ \vdots & \vdots & & \vdots \\ h_{siv/x}[1-N] & h_{siv/x}[2-N] & \cdots & h_{siv/x}[0] \end{bmatrix}$$

- Operation is a 2-d separable convolution („filtering“)

$$g[\alpha,\beta] = \sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_{siv/x}[\alpha - x]\, h_{siv/y}[\beta - y]$$

# Non-separable 2-d convolution

- Convolution kernel of linear shift-invariant system („filter“) can also be non-separable

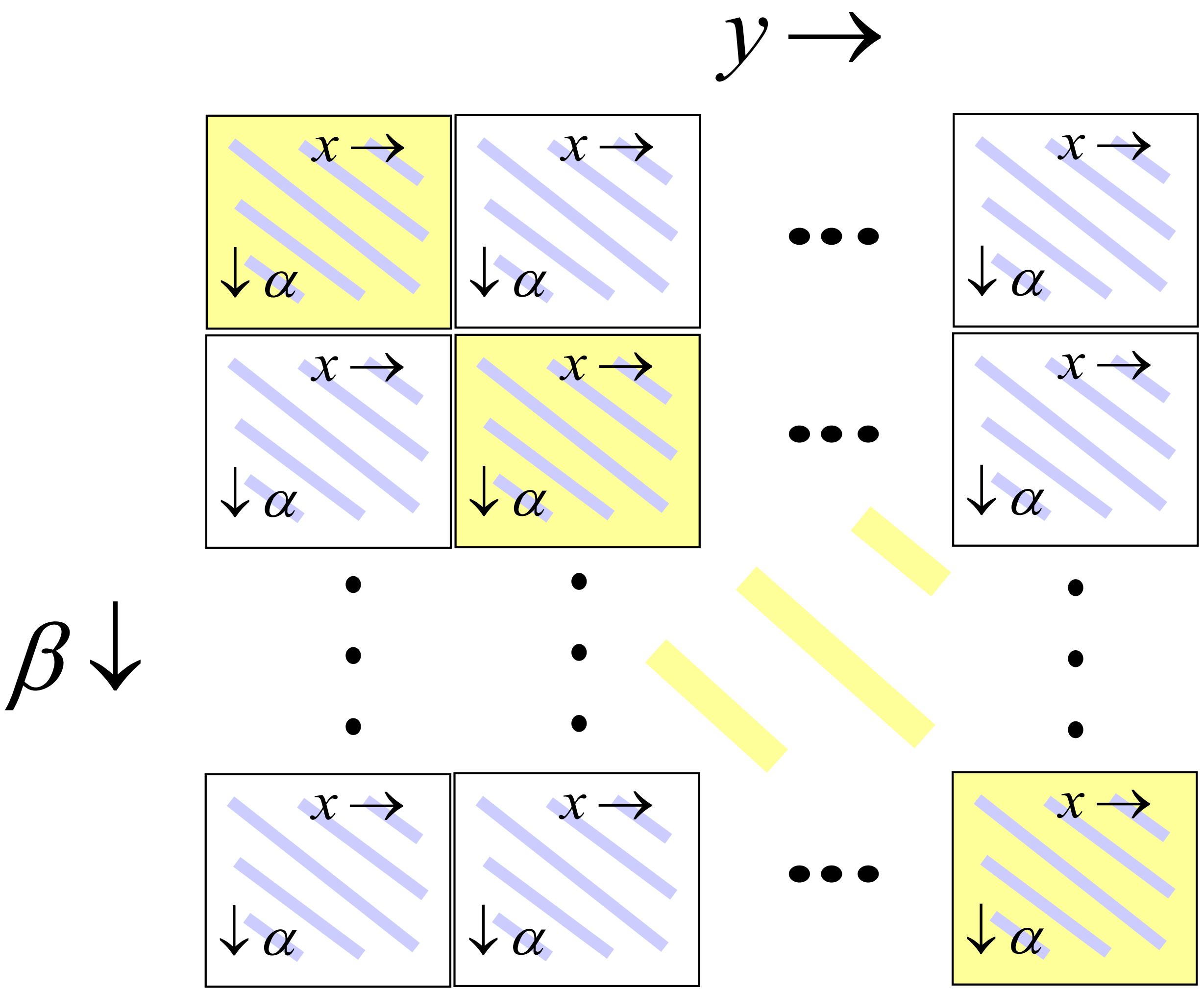
$$g[\alpha,\beta] = \sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_{siv}[\alpha - x, \beta - y]$$

- Viewed as a matrix operation . . .

$$\vec{g} = H\vec{f}$$

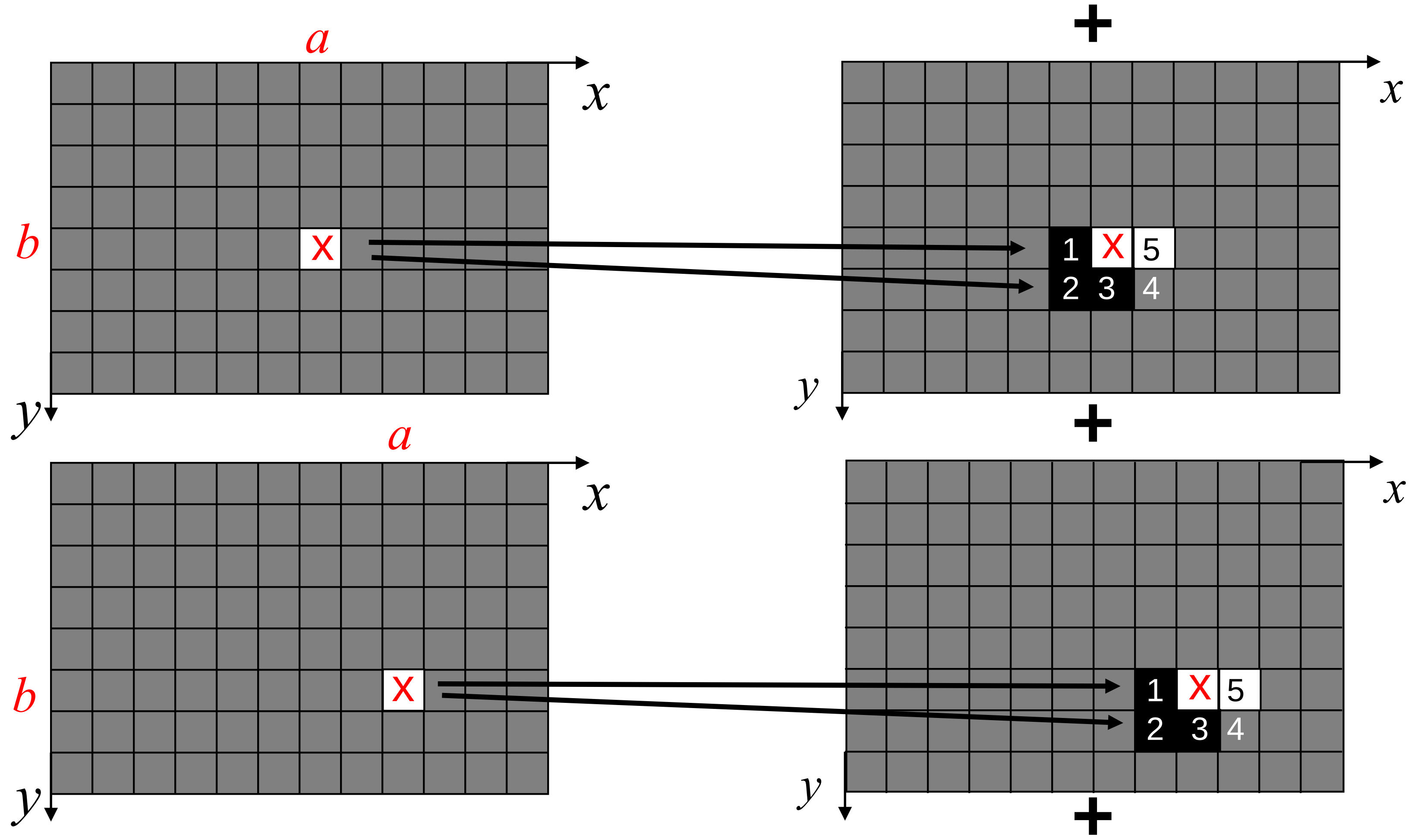
. . . $\boldsymbol{H}$ is a block Toeplitz matrix

# Structure of $H$ for non-separable convolution



# Convolution: superposition of impulse responses

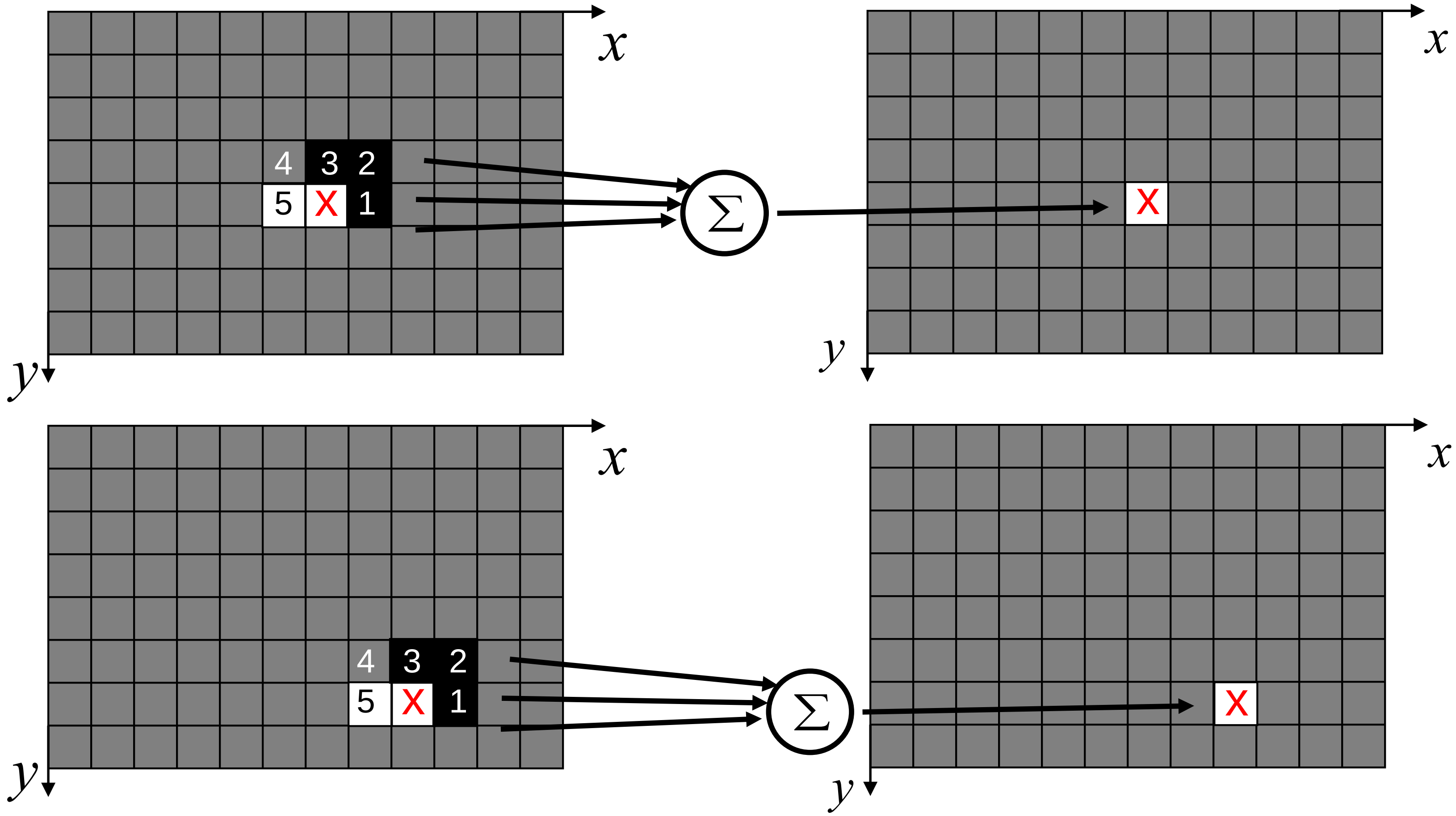
$$f[a,b]\cdot\delta[x-a,y-b] \qquad\qquad g[x,y]=h_{siv}[x-a,y-b]\,f[a,b]$$



# Convolution: linear combination of neighboring pixel values

$$f[x,y]\cdot h_{siv}[\alpha - x, \beta - y]$$

$$g[\alpha,\beta] = \sum_{x=0}^{N-1}\sum_{y=0}^{L-1} f[x,y]\cdot h_{siv}[\alpha - x, \beta - y]$$



# Convolution examples

Original
*Bike*

*Bike* blurred by convolution
Impulse response „box filter“

$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

# Convolution examples

Original

*Bike*

*Bike* blurred horizontally

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & [1] & 1 & 1 \end{pmatrix}$$

# Convolution examples

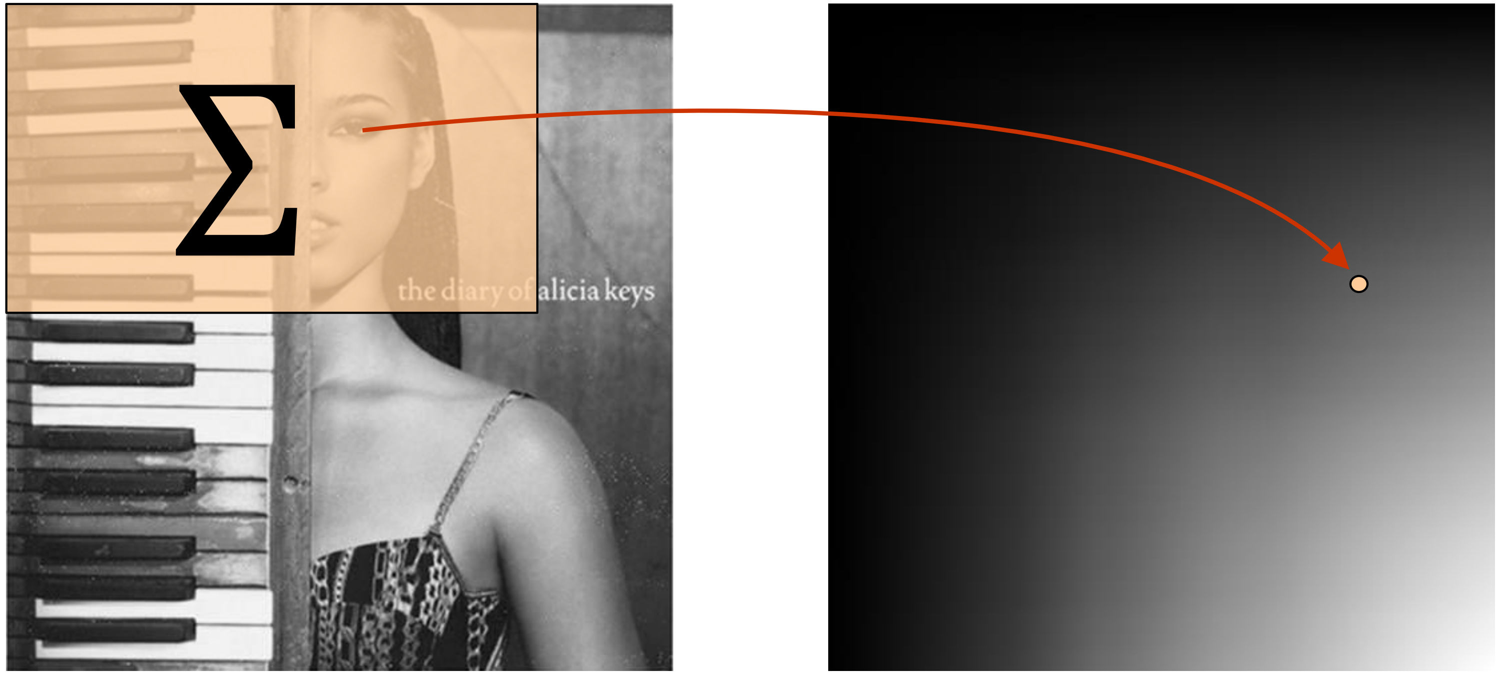
Original
*Bike*

*Bike* blurred vertically

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$

# Integral image

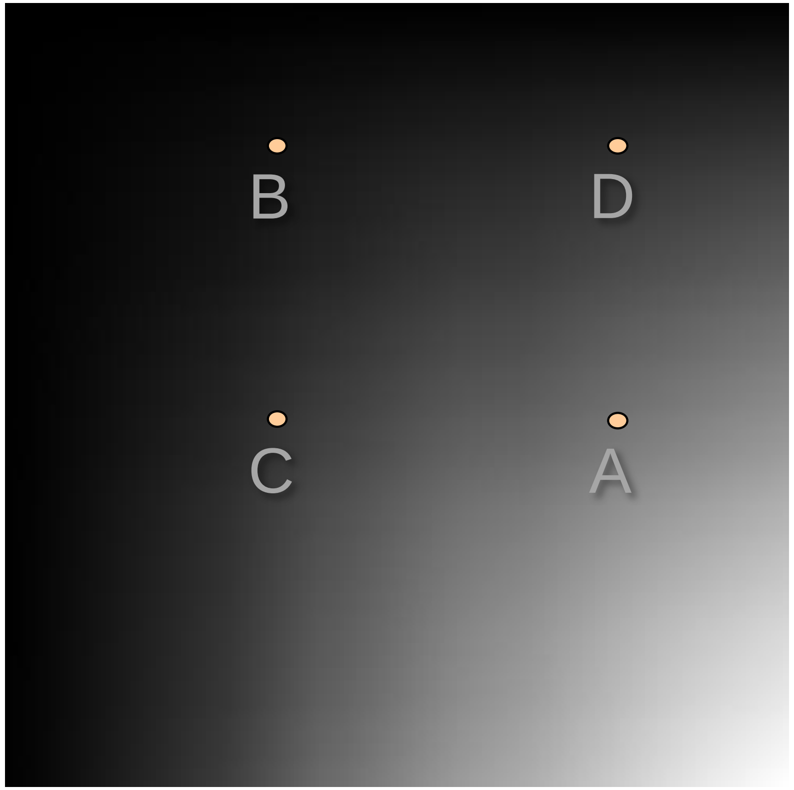



$$j[x,y] = j[x,y-1] + f[x,y]$$
$$i[x,y] = i[x-1,y] + j[x,y]$$

$$f[x,y]$$

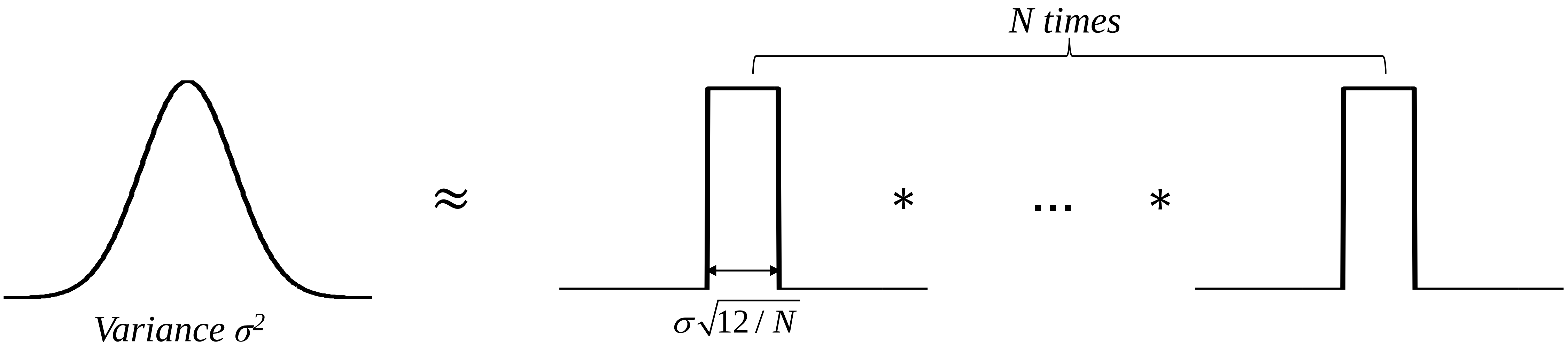
$$i[x,y] = \sum_{u=0}^{x} \sum_{v=0}^{y} f[u,v]$$

# Box filtering with integral image

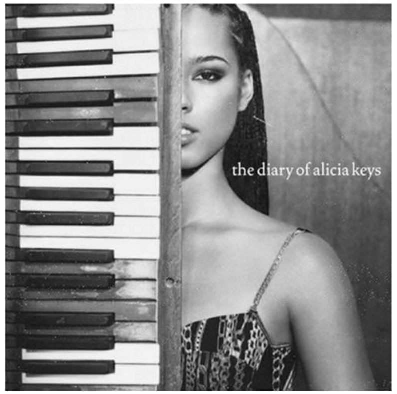

Sum = A + B – C – D

# Gaussian filtering by repeated box filtering




- Approximate convolution with Gaussian kernel by convolution with box kernel, repeated N times
- Convolution with box kernel can be computed efficiently using integral image

# Gaussian filtering by repeated box filtering



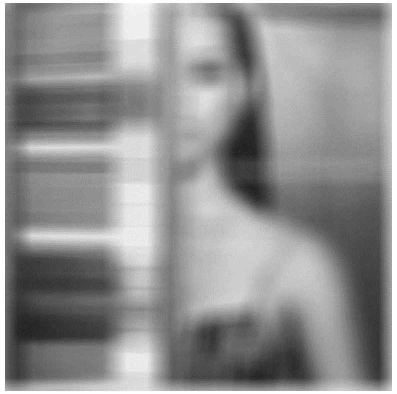

Original image
500x500

Gaussian filtered
σ = 20, 81x81 kernel

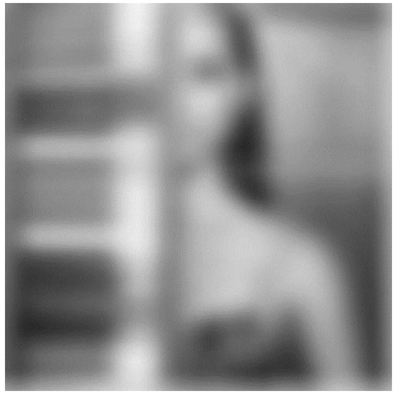
Direct implementation:
6561 multiplications and
6560 additions per pixel

x-y separable filtering:
162 multiplications and
160 additions per pixel

20 additions or subtractions per pixel



Box filtered
after N = 1



Box filtered
after N = 2

Box filtered
after N = 3

Box filtered
after N = 4

# 1-d discrete-time Fourier transform

- Given a 1-d sequence $s[k], \quad k \in \mathbf{Z} = \{\ldots, -1, 0, 1, 2, 3, \ldots\}$
- Fourier transform

$$S\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} s[k] e^{-j\omega k} \qquad \omega \in \Re$$

- Fourier transform is periodic with $2\pi$
- Inverse Fourier transform

$$s[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S\left(e^{j\omega}\right) e^{j\omega k} d\omega$$

# 2-d discrete-space Fourier transform

- Given a 2-d array of image samples

$$s[m,n], \quad m,n \in \mathbf{Z}^2$$

- Fourier transform

$$S\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{-j\omega_x m - j\omega_y n} \qquad \omega_x, \omega_y \in \Re^2$$

- Fourier transform is $2\pi$-periodic both in $\omega_x$ and $\omega_y$
- Inverse Fourier transform

$$s[m,n] = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S\left(e^{j\omega_x}, e^{j\omega_y}\right) e^{j\omega_x m + j\omega_y n} d\omega_x d\omega_y$$

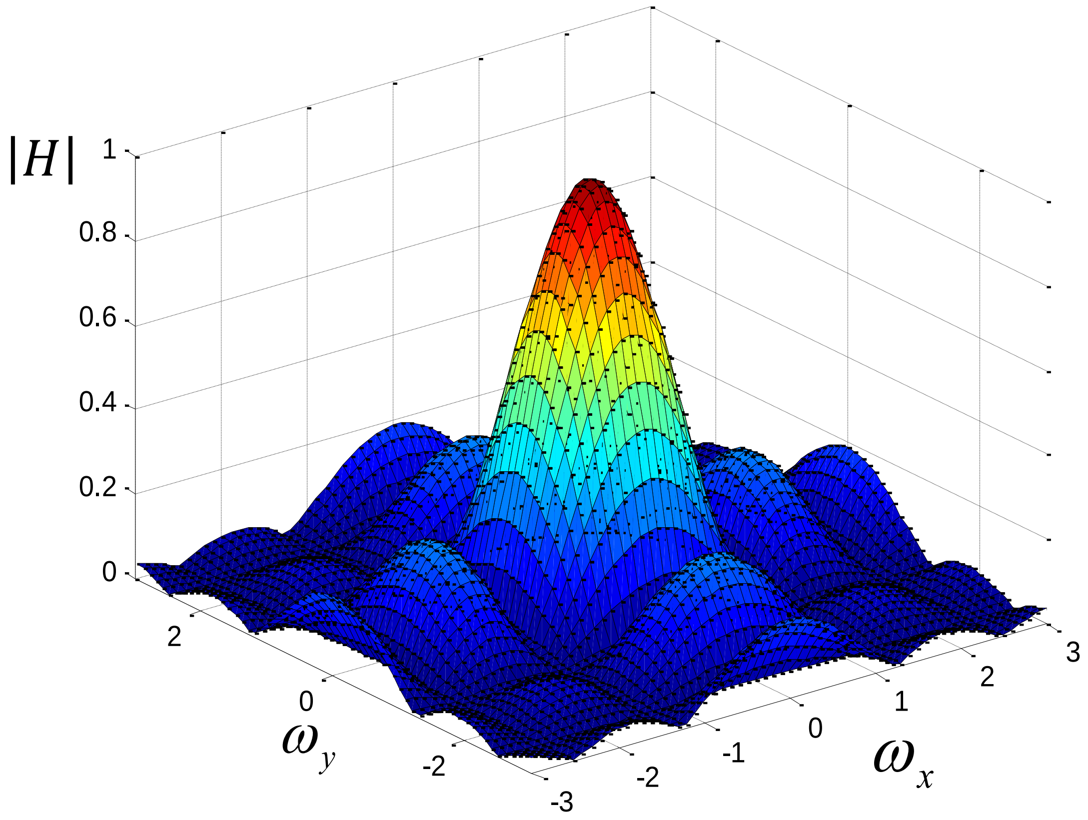
# 5x5 box filter revisited

Original
*Bike*

*Bike* blurred by convolution
Impulse response „box filter“

$$\frac{1}{25}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

# Frequency response of 5x5 lowpass filter




$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m,n] e^{-j\omega_x m - j\omega_y n}$$

$$= \frac{1}{25} \sum_{m=-2}^{2} \sum_{n=-2}^{2} e^{-j\omega_x m - j\omega_y n} = \frac{1}{25} \sum_{m=-2}^{2} e^{-j\omega_x m} \sum_{n=-2}^{2} e^{-j\omega_y n}$$

$$= \frac{1}{25} \left(1 + 2\cos\omega_x + 2\cos(2\omega_x)\right)\left(1 + 2\cos\omega_y + 2\cos(2\omega_y)\right)$$

Separable filter:
1-d frequency responses are multiplied

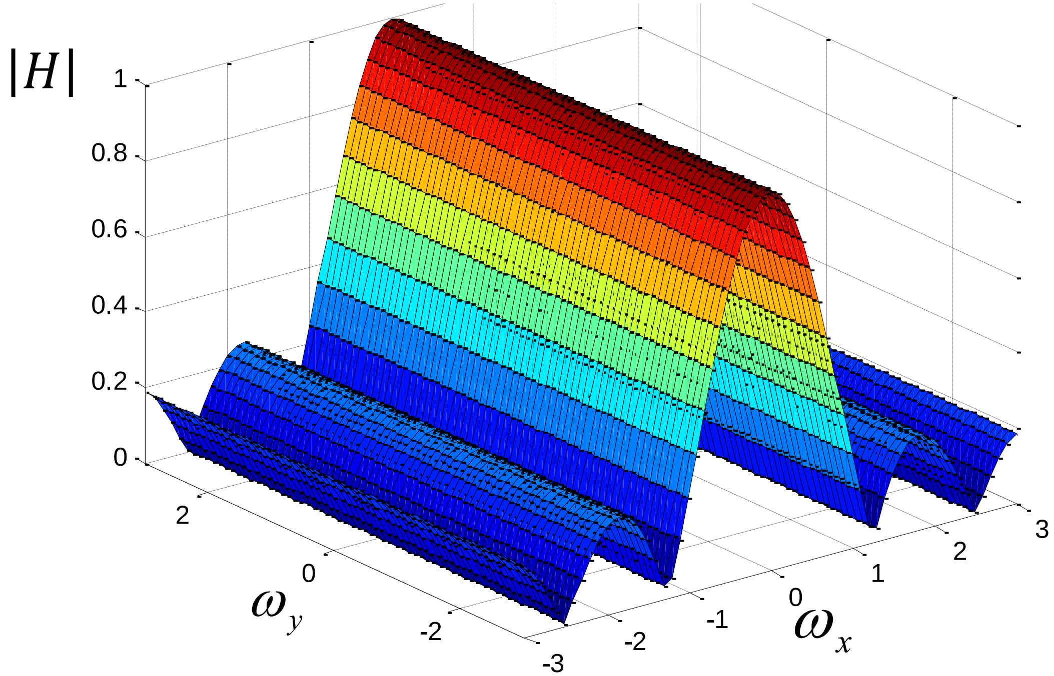
# Horizontal lowpass filter

*Bike* blurred horizontally

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & \boxed{1} & 1 & 1 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos\omega_x + 2\cos(2\omega_x)\right)$$



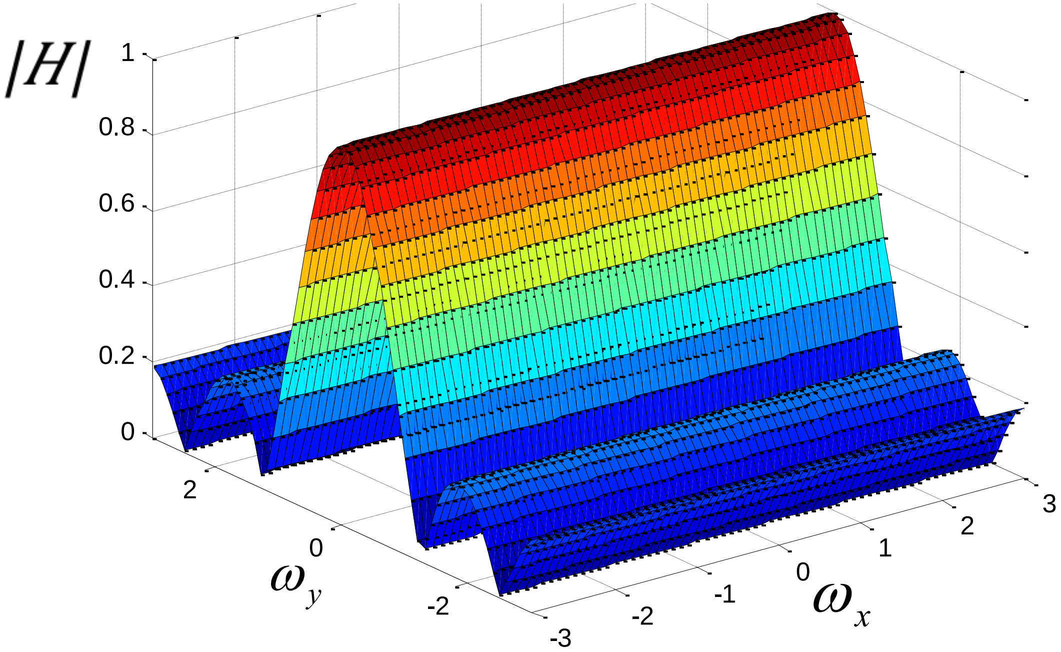
# Vertical lowpass filter

*Bike* blurred vertically

Filter impulse response

$$\frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{5}\left(1 + 2\cos\omega_y + 2\cos(2\omega_y)\right)$$



# Sharpening filter

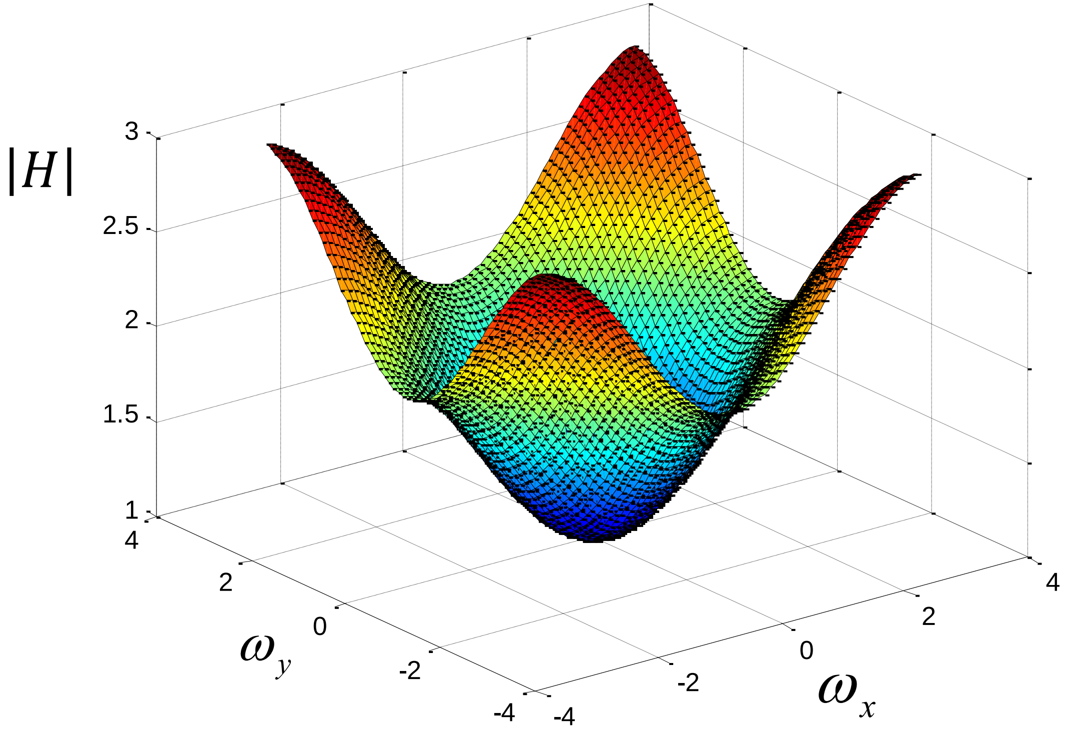
Original
*Bike*

*Bike* sharpened

Filter impulse response

$$\frac{1}{4}\begin{pmatrix} 0 & -1 & 0 \\ -1 & [8] & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

# Frequency response of sharpening filter




$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m,n] e^{-j\omega_x m - j\omega_y n}$$

$$= \frac{1}{4}\left(8 - e^{-j\omega_x} - e^{j\omega_x} - e^{-j\omega_y} - e^{j\omega_y}\right)$$

$$= 2 - \frac{1}{2} \cos\omega_x - \frac{1}{2} \cos\omega_y$$

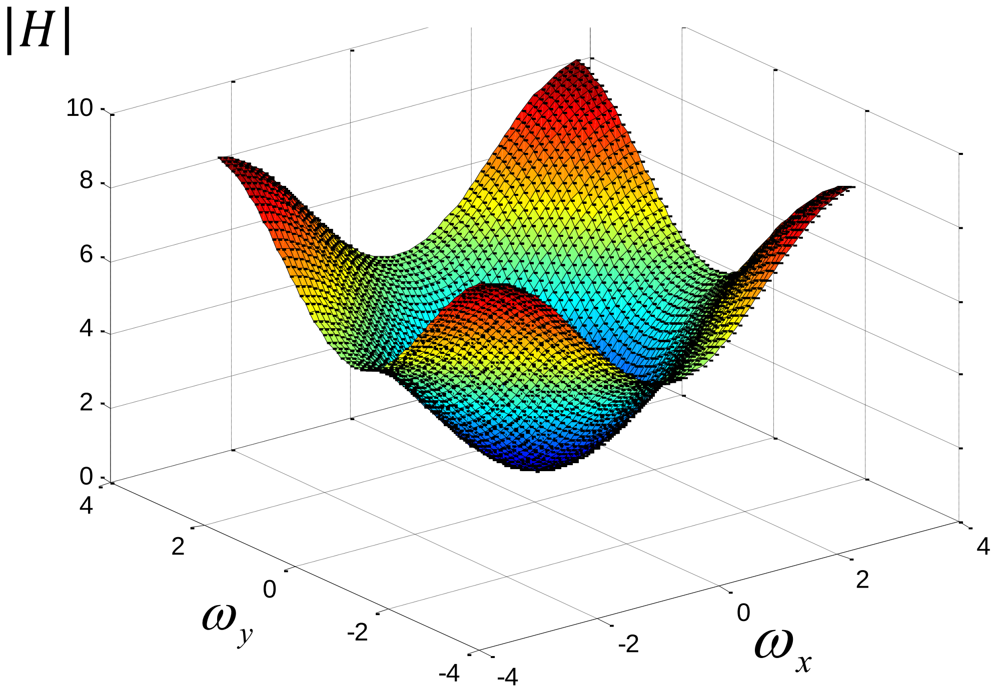
# More aggressive sharpening

*Bike* sharpened

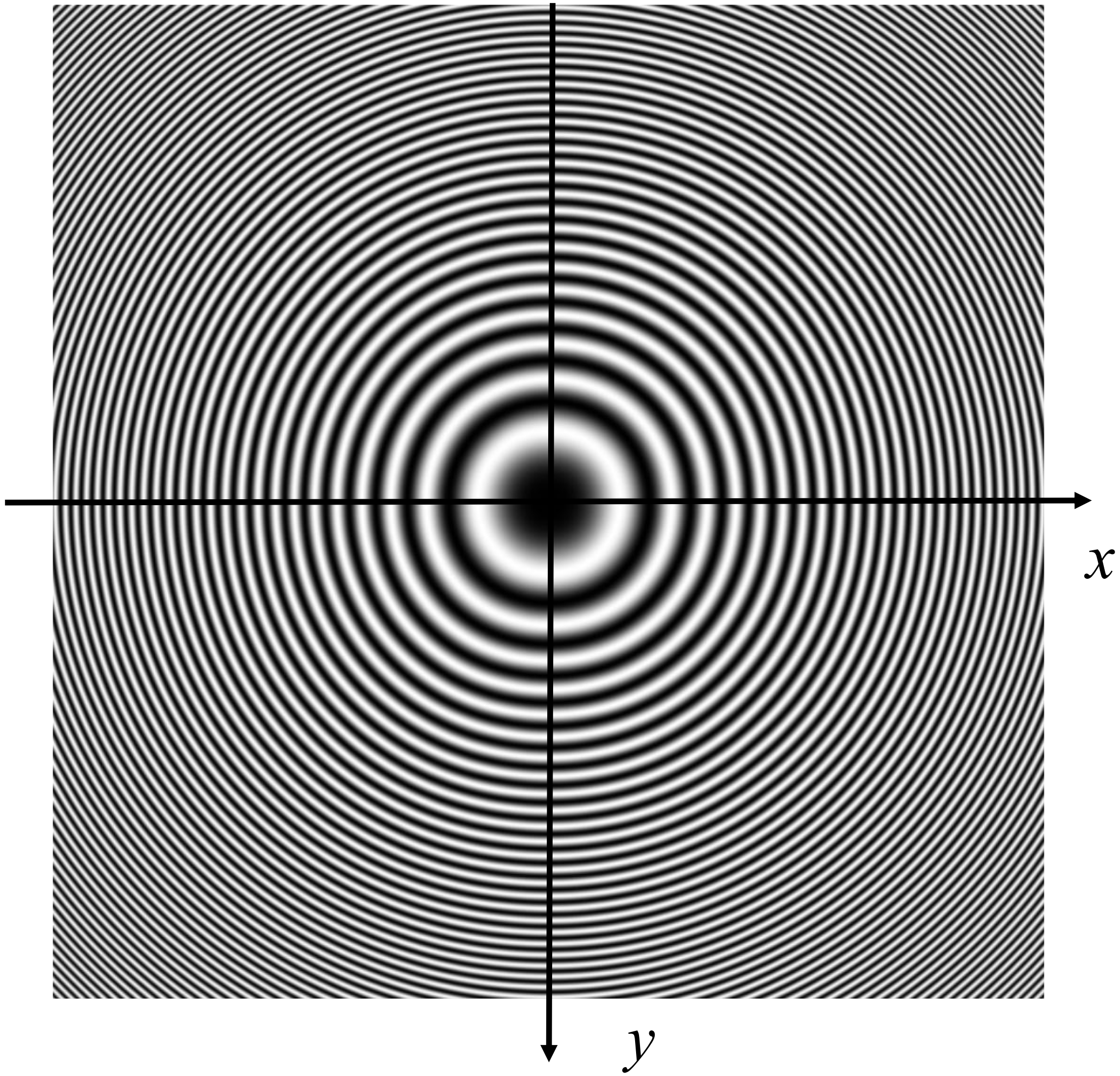
Filter impulse response

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & [5] & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = 5 - 2\cos\omega_x - 2\cos\omega_y$$



# Zoneplate pattern to visualize frequency plane




Equation to generate pattern:

$$s(x,y) = \hat{s}\cos(a_x x^2 + a_y y^2) + s_0$$

Local frequency at $(x, y)$

$$\frac{\partial}{\partial x}\left(a_x x^2 + a_y y^2\right) = 2a_x x$$

$$\frac{\partial}{\partial y}\left(a_x x^2 + a_y y^2\right) = 2a_y y$$

$$s[x,y] \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} s(x,y)$$

# Sampling interpretation of 2-d discrete-space Fourier transform

- How is the Fourier transform of $s[m,n]$ related to the Fourier transform of the continuous signal

$$s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n]\, \delta_2(x-m, y-n)$$

continuous

discrete

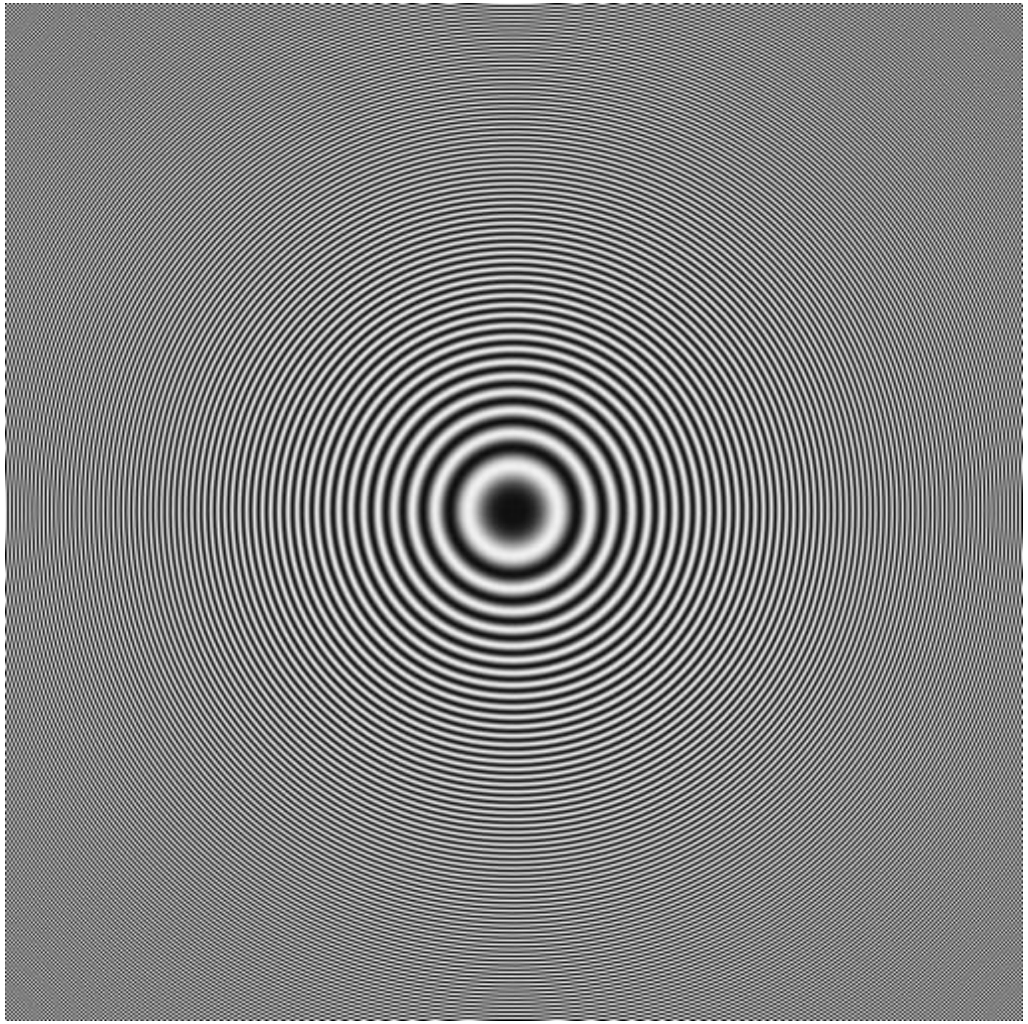
„continuous“
delta function

- Continuous-space 2-d Fourier transform

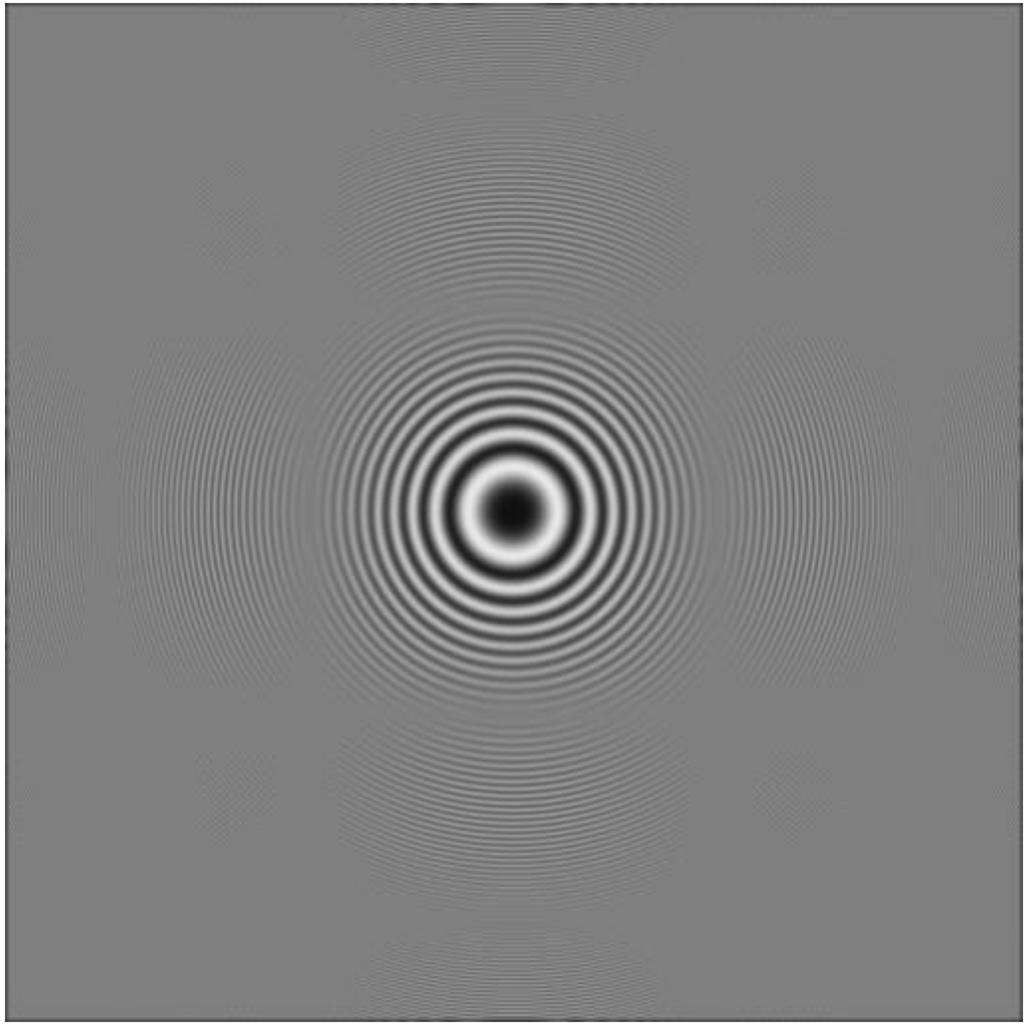
$$\hat{S}(\omega_x, \omega_y) = \int_x \int_y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n]\, \delta_2(x-m, y-n)\, e^{-j\omega_x x - j\omega_y y} dxdy$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{-j\omega_x m - j\omega_y n} = S\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

# Lowpass filtering

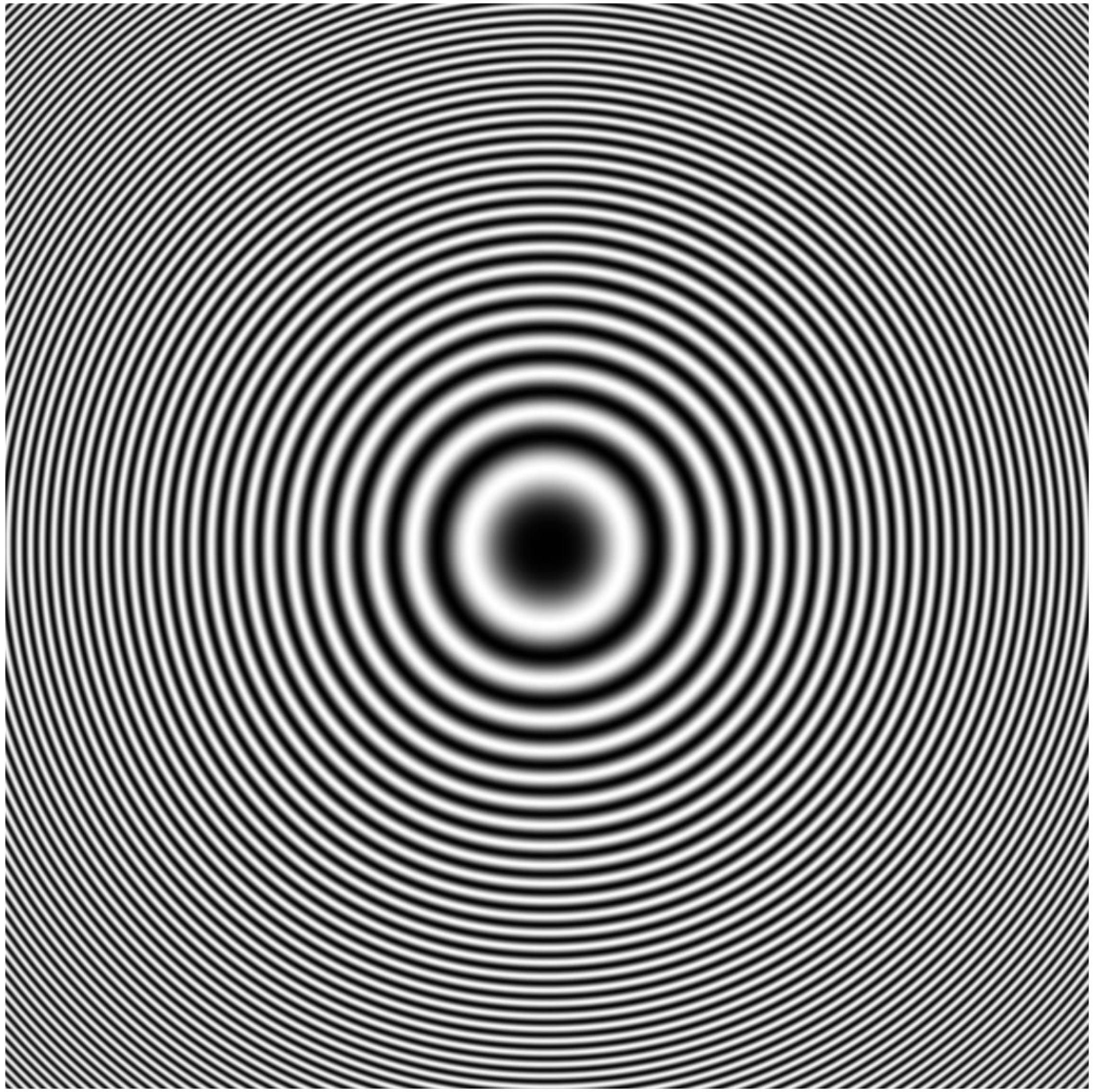


Original *Zoneplate* (512x512)

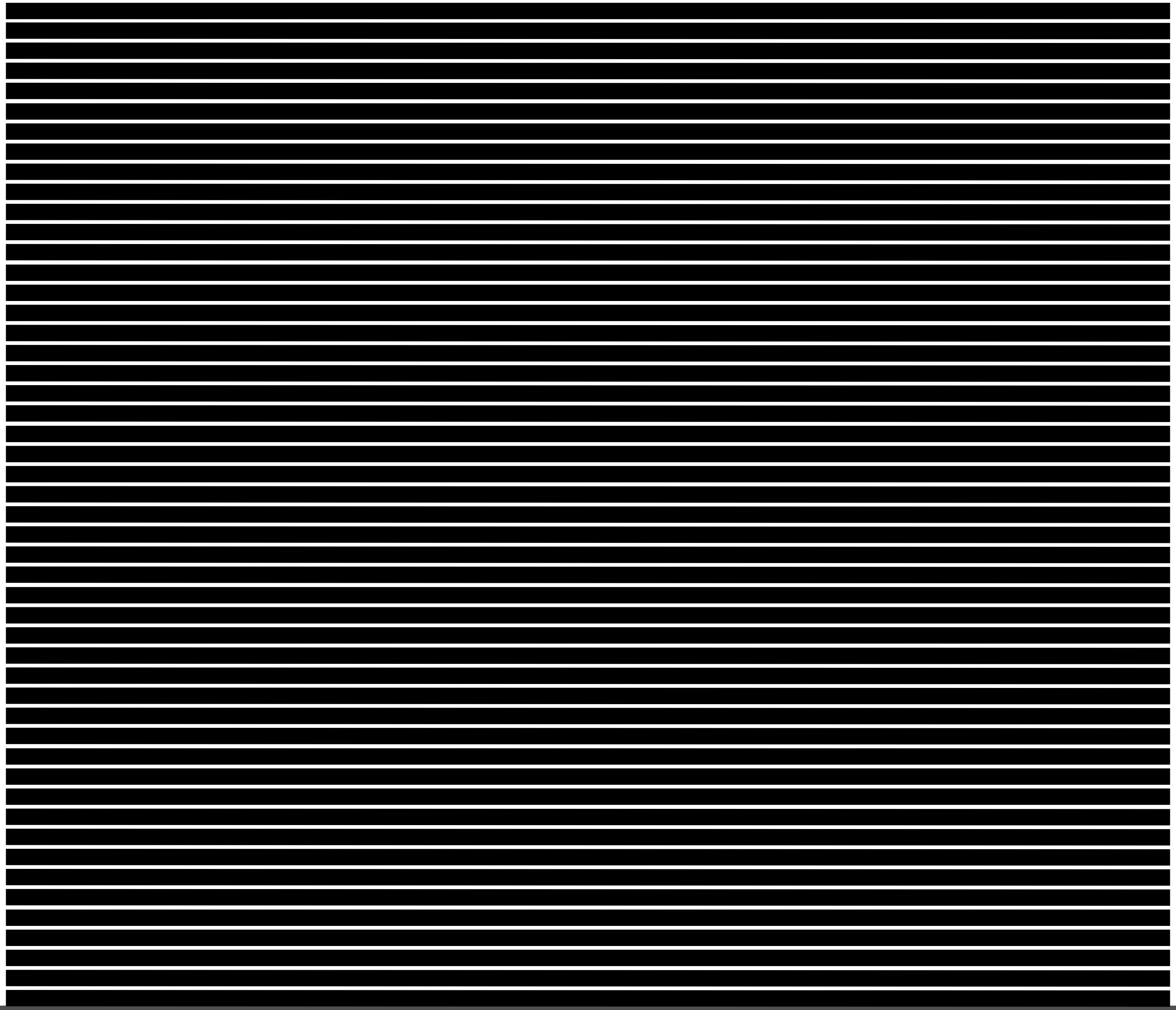


Lowpass filtered with 5x5 box filter

# Zoneplate demonstration: 1-d sampling of 2-d signals



# Zoneplate demonstration: 1-d sampling of 2-d signals



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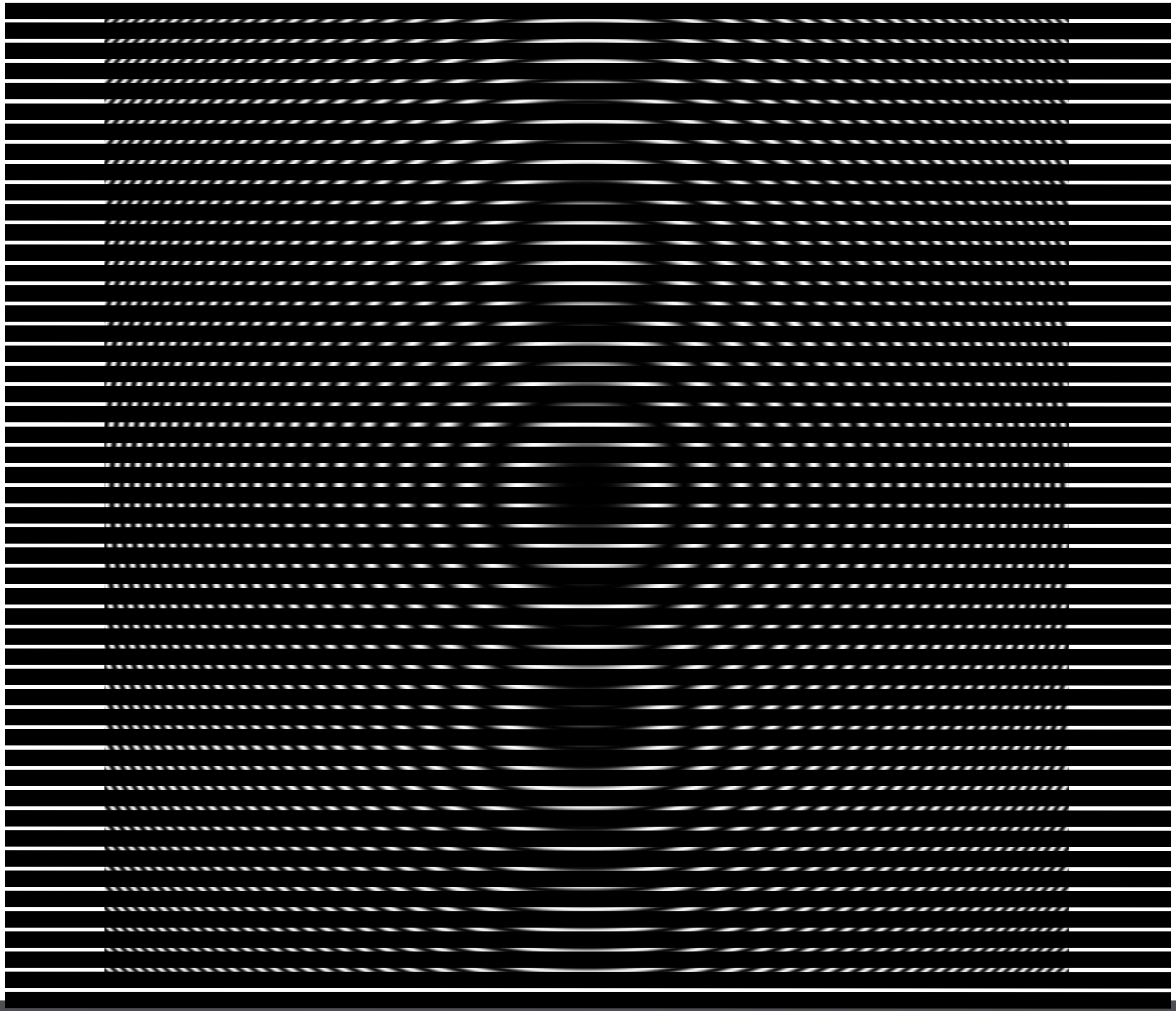
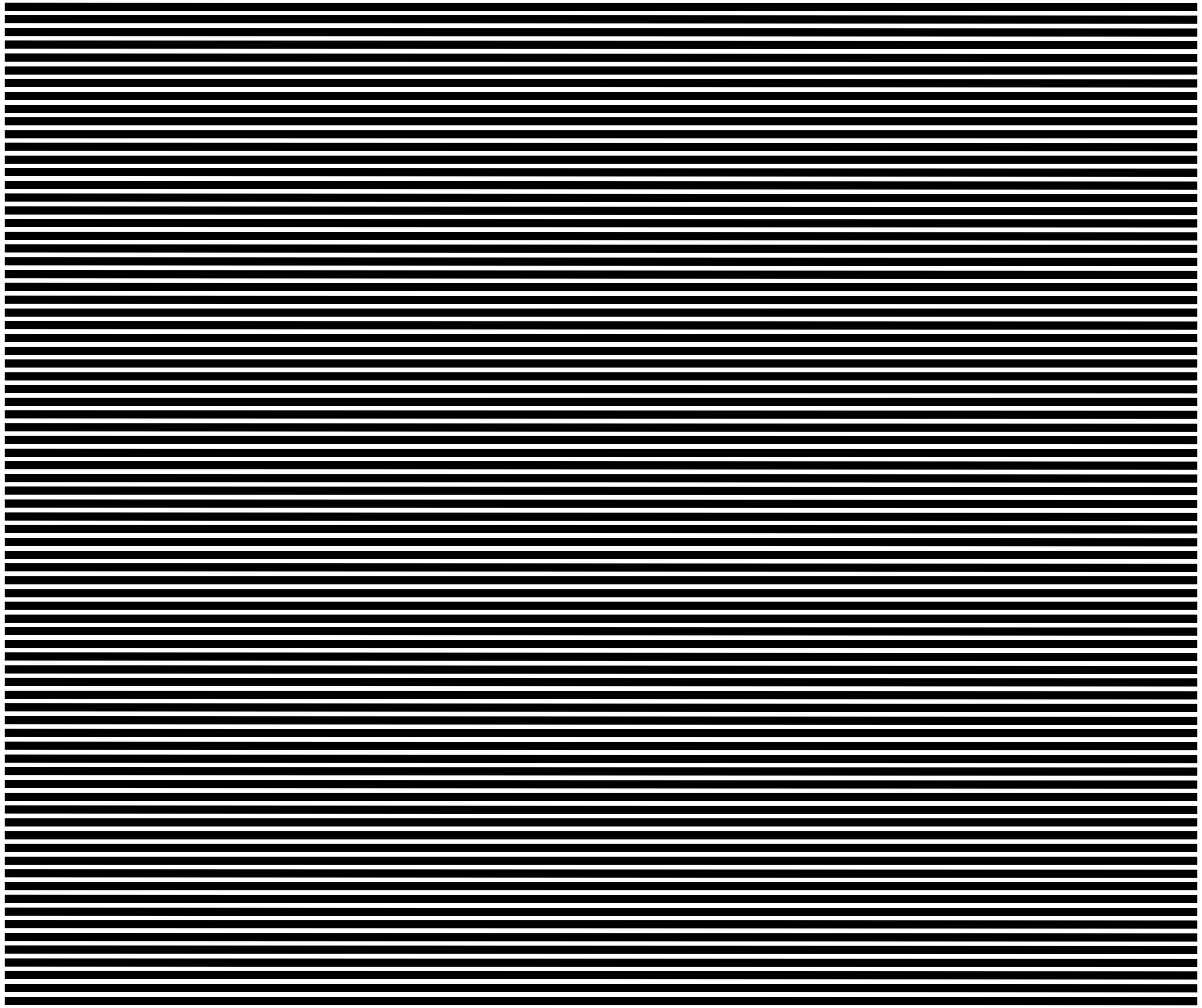
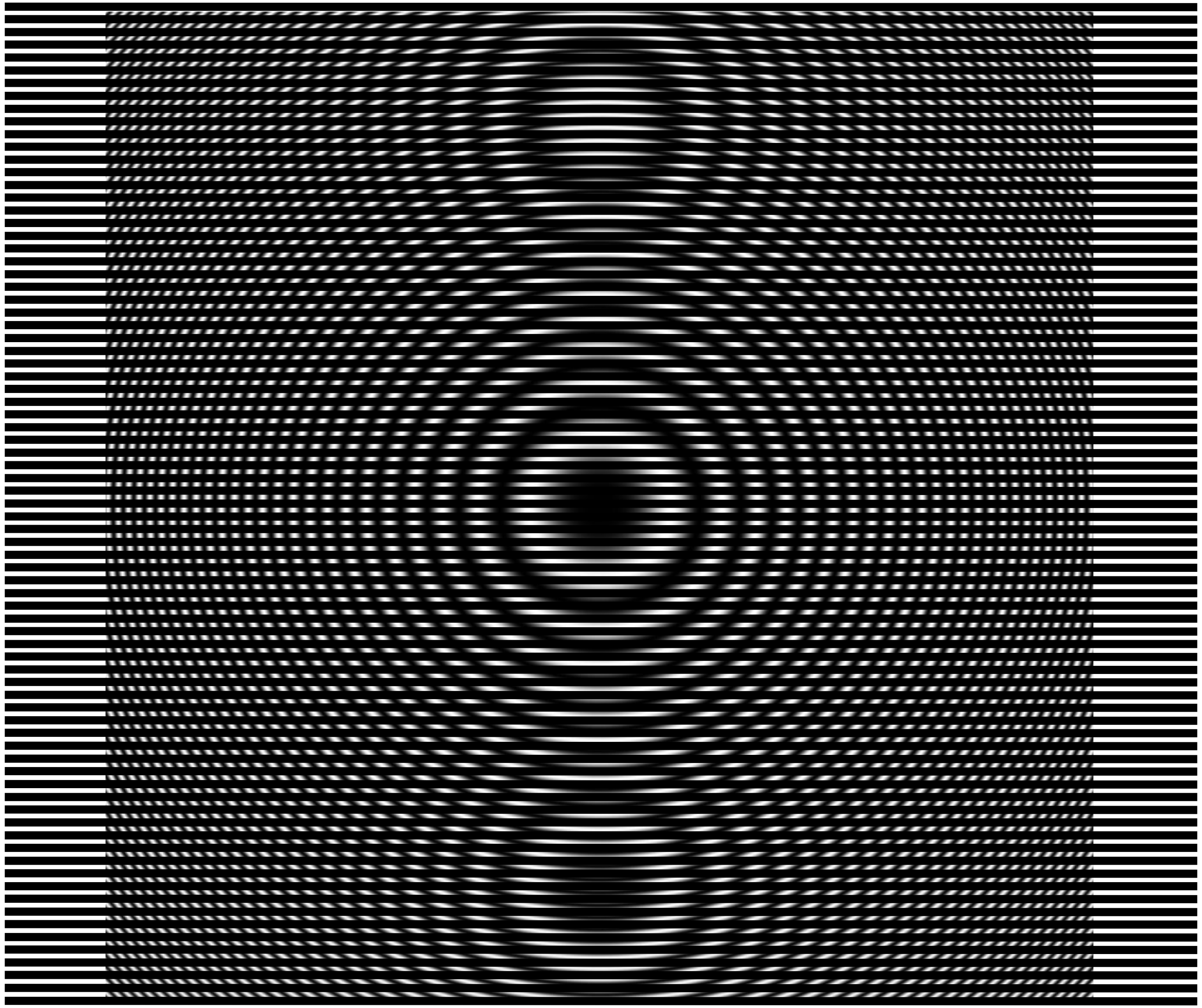


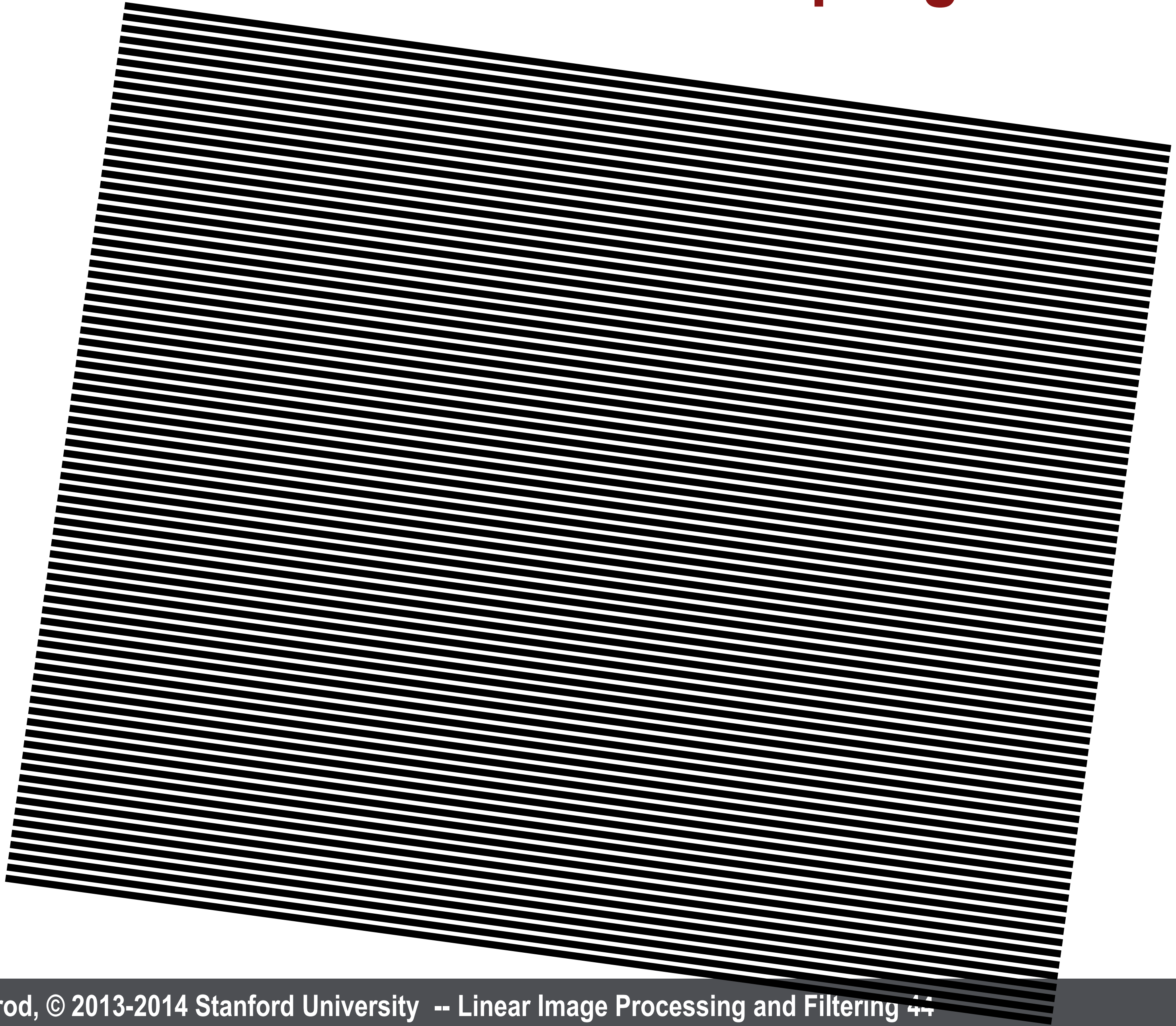
# Zoneplate demonstration: 1-d sampling of 2-d signals



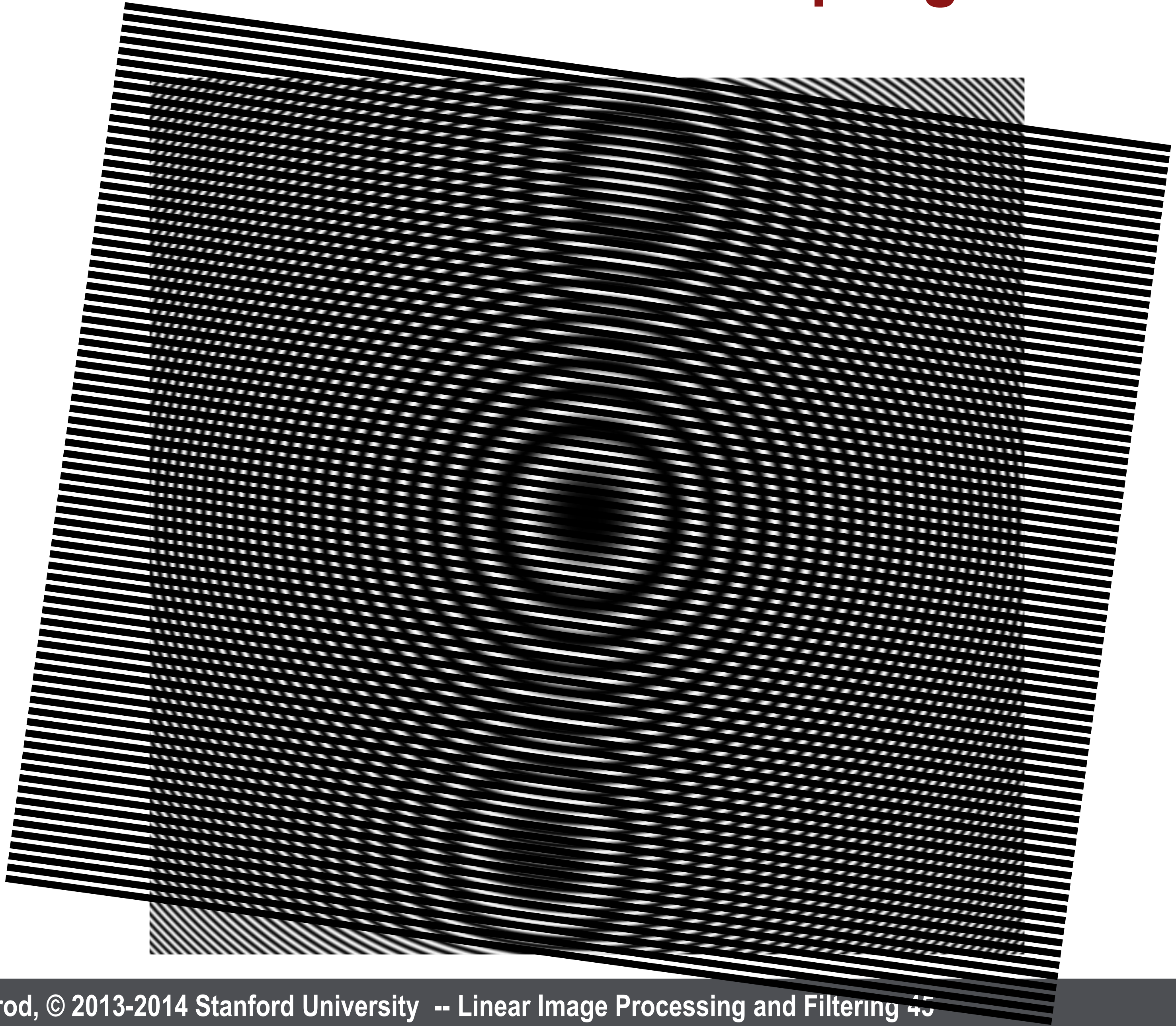
# Zoneplate demonstration: 1-d sampling of 2-d signals



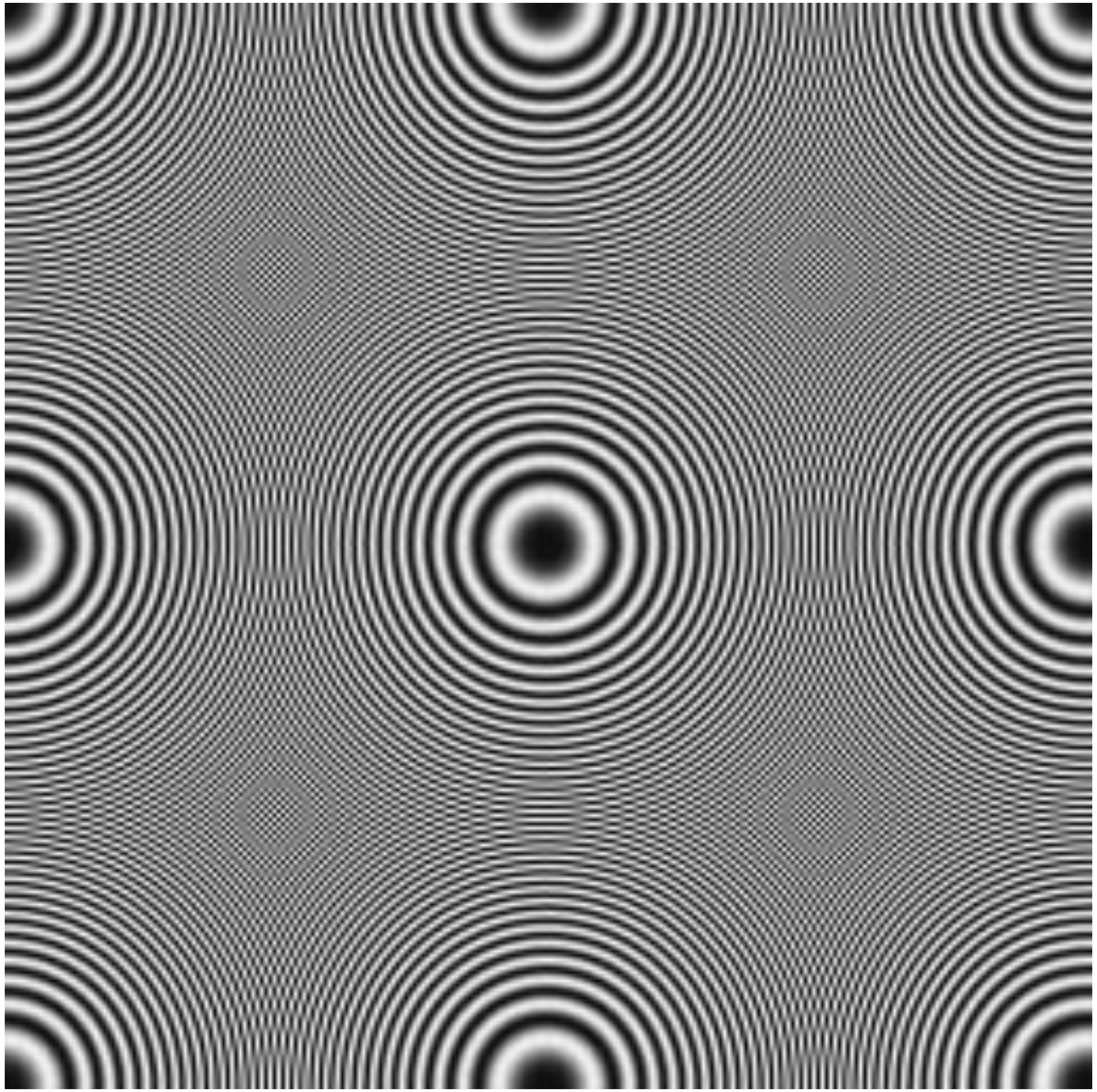
# Zoneplate demonstration: 1-d sampling of 2-d signals



# Zoneplate demonstration: 1-d sampling of 2-d signals



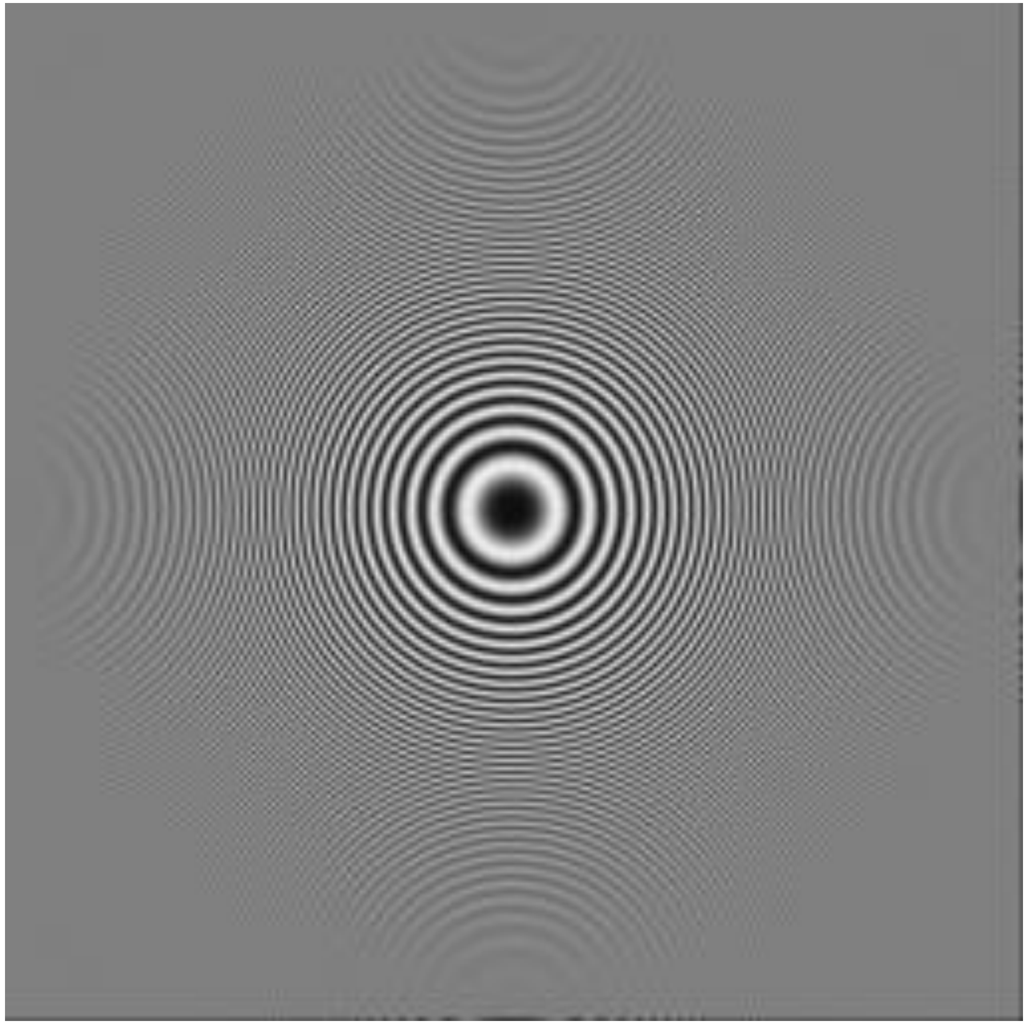
# Horizontal/vertical 2:1 subsampling without prefiltering



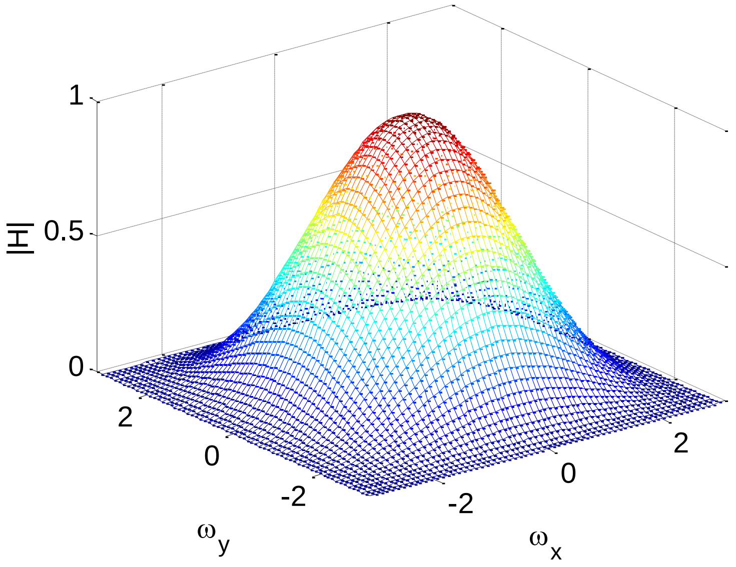
# Horizontal/vertical 2:1 subsampling with prefiltering

<u>2d impulse response</u>

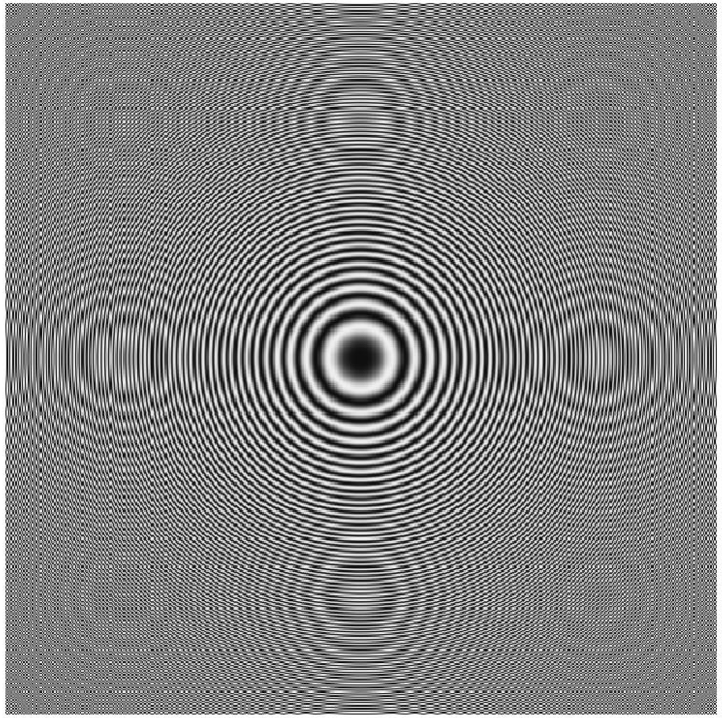
$$\begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$



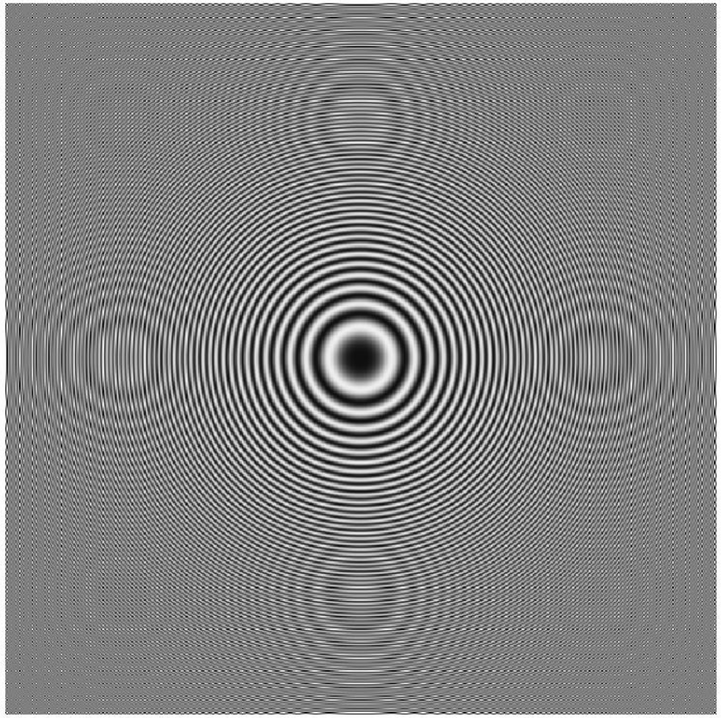
<u>Frequency response</u>



# Image magnification (10%)



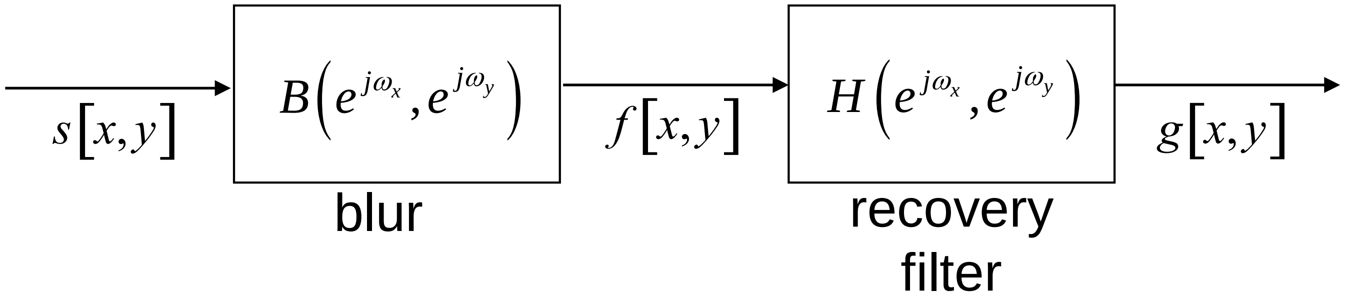
Nearest neighbor interpolation



Bilinear interpolation

# Image deconvolution

- Given an image $f[x,y]$ that is a blurred version of original image $s[x,y]$, recover the original image.
- Assume linear shift-invariant blur, transfer function $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$




- Naive solution: inverse filter

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{1}{B\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$

- Problem: $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$ might be zero, noise amplification

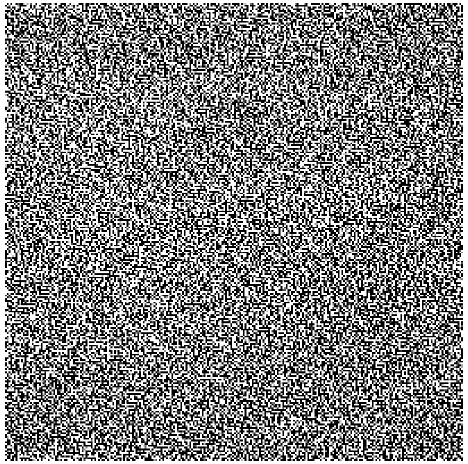
# Naïve deconvolution

Original

Blurred w/ $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$

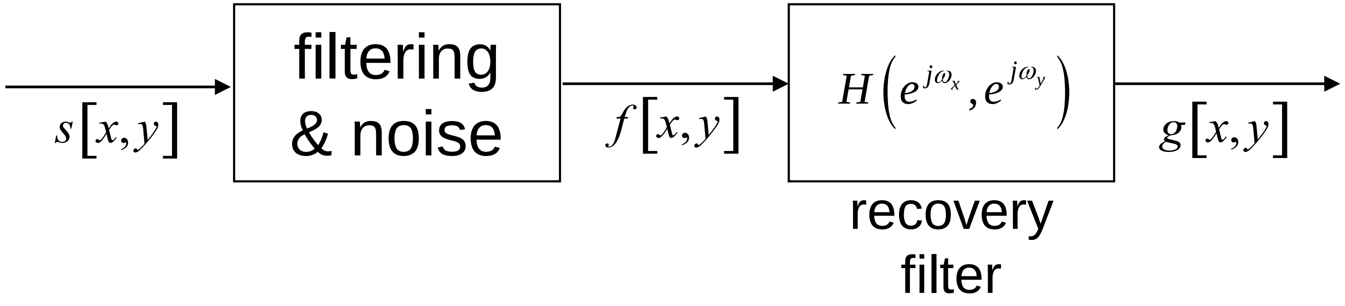
Deblurred w/ $B^{-1}\left(e^{j\omega_x}, e^{j\omega_y}\right)$

+ white noise $rms = 2.5$



# Wiener filtering

- Model




- Minimize mean squared estimation error

$$E\left\{e^2[x,y]\right\} = E\left\{\left(g[x,y]-s[x,y]\right)^2\right\} \xrightarrow{H\left(e^{j\omega_x},e^{j\omega_y}\right)} \min.$$

- Power spectral density of estimation error

$$\begin{aligned}\Phi_{ee}\left(e^{j\omega_x},e^{j\omega_y}\right) &= \Phi_{gg}\left(e^{j\omega_x},e^{j\omega_y}\right) - \Phi_{gs}\left(e^{j\omega_x},e^{j\omega_y}\right) - \Phi_{sg}\left(e^{j\omega_x},e^{j\omega_y}\right) + \Phi_{ss}\left(e^{j\omega_x},e^{j\omega_y}\right) \\ &= \Phi_{ff}\left(e^{j\omega_x},e^{j\omega_y}\right) H\left(e^{j\omega_x},e^{j\omega_y}\right) H^*\left(e^{j\omega_x},e^{j\omega_y}\right) - \Phi_{fs}\left(e^{j\omega_x},e^{j\omega_y}\right) H\left(e^{j\omega_x},e^{j\omega_y}\right) \\ &\quad - \Phi_{sf}\left(e^{j\omega_x},e^{j\omega_y}\right) H^*\left(e^{j\omega_x},e^{j\omega_y}\right) + \Phi_{ss}\left(e^{j\omega_x},e^{j\omega_y}\right)\end{aligned}$$

# Review: Power spectrum and cross spectrum

- 2-d discrete-space cross correlation function for ergodic, stationary signals

$$\varphi_{fg}[m,n] = E\left\{ f[x+m, y+n]\, g^*[x,y] \right\}$$

- Special case: autocorrelation function

$$\varphi_{ff}[m,n] = E\left\{ f[x+m, y+n]\, f^*[x,y] \right\}$$

- Cross spectral density

$$\Phi_{fg}\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \varphi_{fg}[m,n]\, e^{-j\omega_x m - j\omega_y n}$$

- Power spectral density

$$\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \varphi_{ff}[m,n]\, e^{-j\omega_x m - j\omega_y n}$$

# Wiener filtering (cont.)

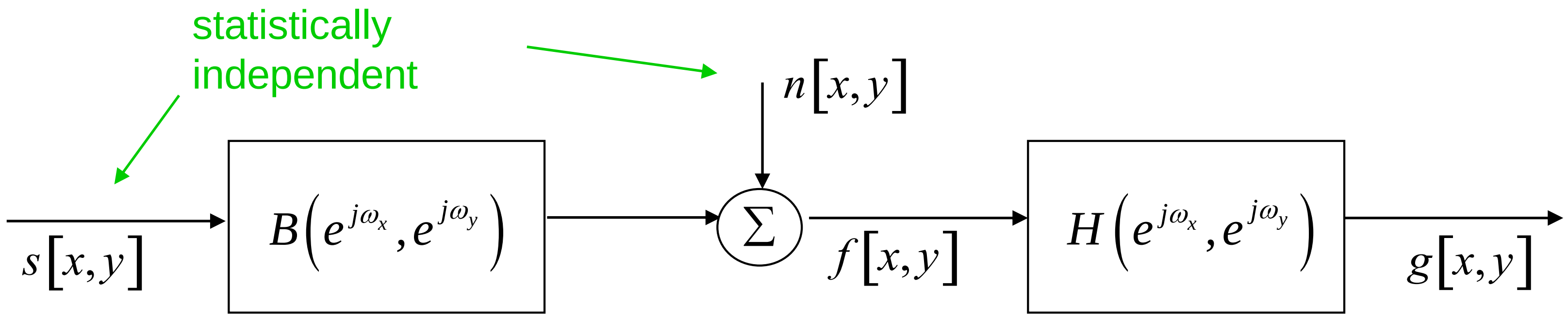
- Power spectrum $\Phi_{ee}$ is minimized separately at each frequency $\omega_x, \omega_y$ if

$$\boxed{H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi^*_{fs}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right)} = \frac{\Phi_{sf}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right)}}$$

- Can be shown to be global minimum by considering filter

$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi^*_{fs}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right)} + \Delta H\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

# Wiener filter for linear distortion and additive noise




$$\Phi_{sf}\left(e^{j\omega_x}, e^{j\omega_y}\right) = \Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) B^*\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

$$\Phi_{ff}\left(e^{j\omega_x}, e^{j\omega_y}\right) = \Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) \left|B\left(e^{j\omega_x}, e^{j\omega_y}\right)\right|^2 + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)$$

$$\boxed{H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) B^*\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) \left|B\left(e^{j\omega_x}, e^{j\omega_y}\right)\right|^2 + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)}}$$

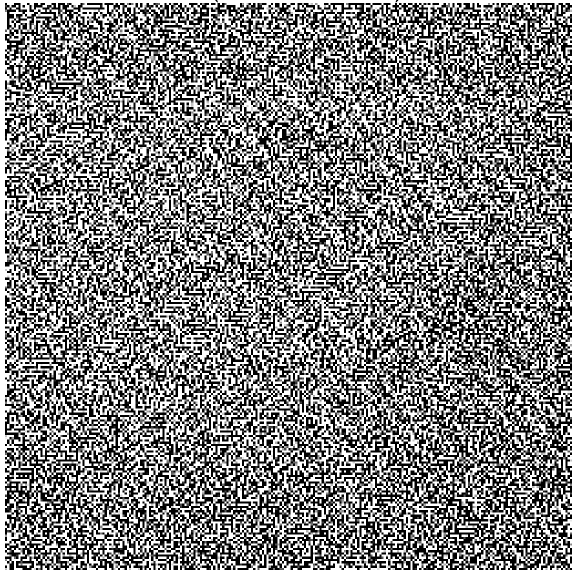
# Wiener filter for linear distortion and additive noise

Blurred w/ $B\left(e^{j\omega_x}, e^{j\omega_y}\right)$

Add. white noise, $rms = 2.5$

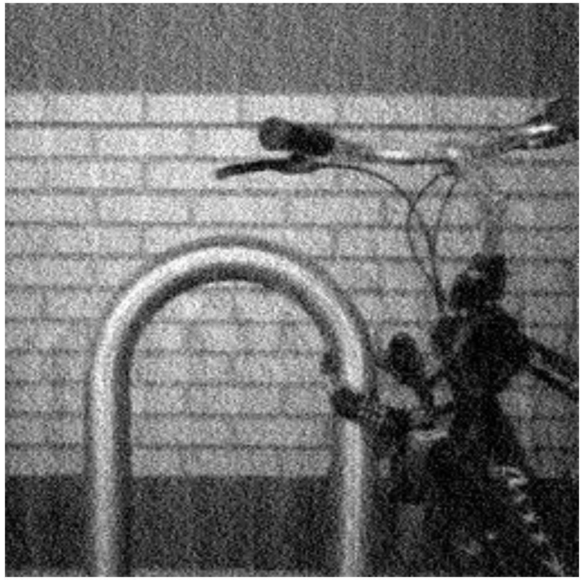
Deblurred

$B^{-1}\left(e^{j\omega_x}, e^{j\omega_y}\right)$

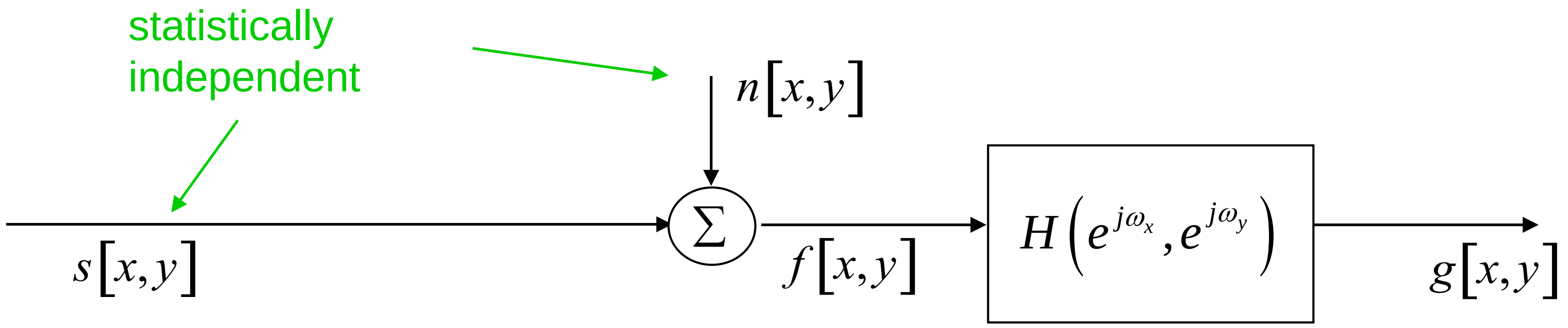


Wiener Filter

$$\frac{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) B^*\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right)\left|B\left(e^{j\omega_x}, e^{j\omega_y}\right)\right|^2 + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$



# Wiener filter for additive noise




$$H\left(e^{j\omega_x}, e^{j\omega_y}\right) = \frac{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right)}{\Phi_{ss}\left(e^{j\omega_x}, e^{j\omega_y}\right) + \Phi_{nn}\left(e^{j\omega_x}, e^{j\omega_y}\right)}$$

# Wiener filter for additive noise

Original

Additive
white noise

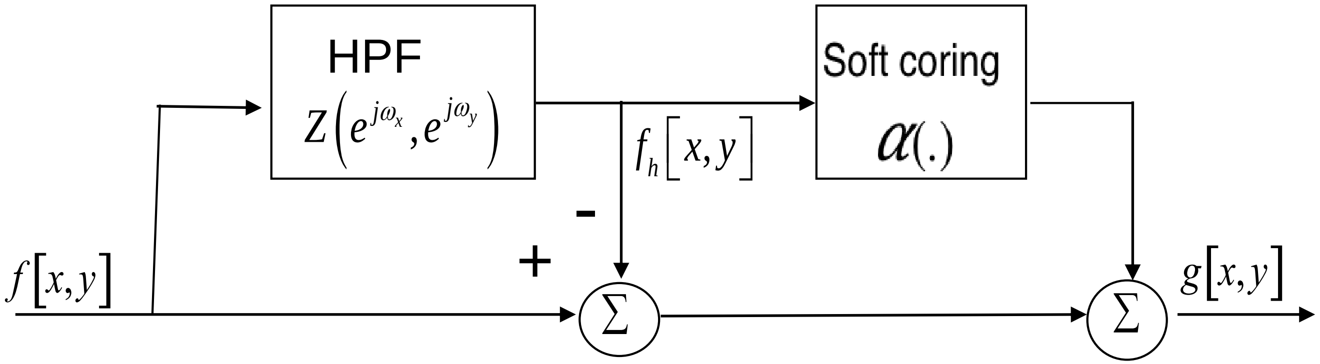
$rms = 17.7$

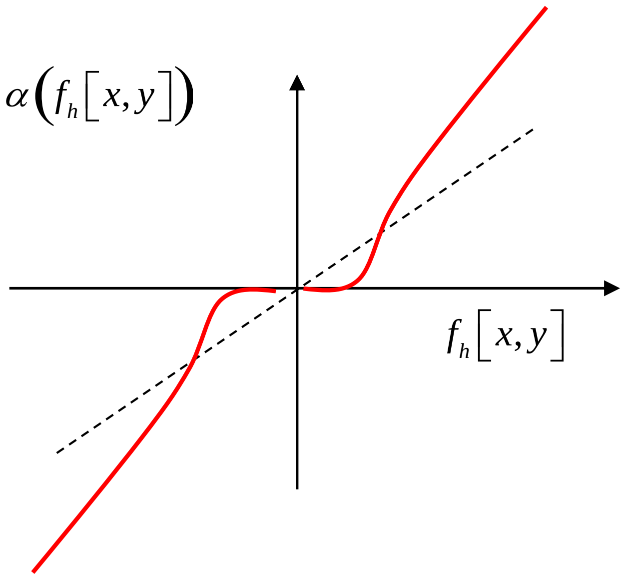
Noise reduction
by Wiener filtering

$rms = 8.8$

# Nonlinear noise reduction/sharpening

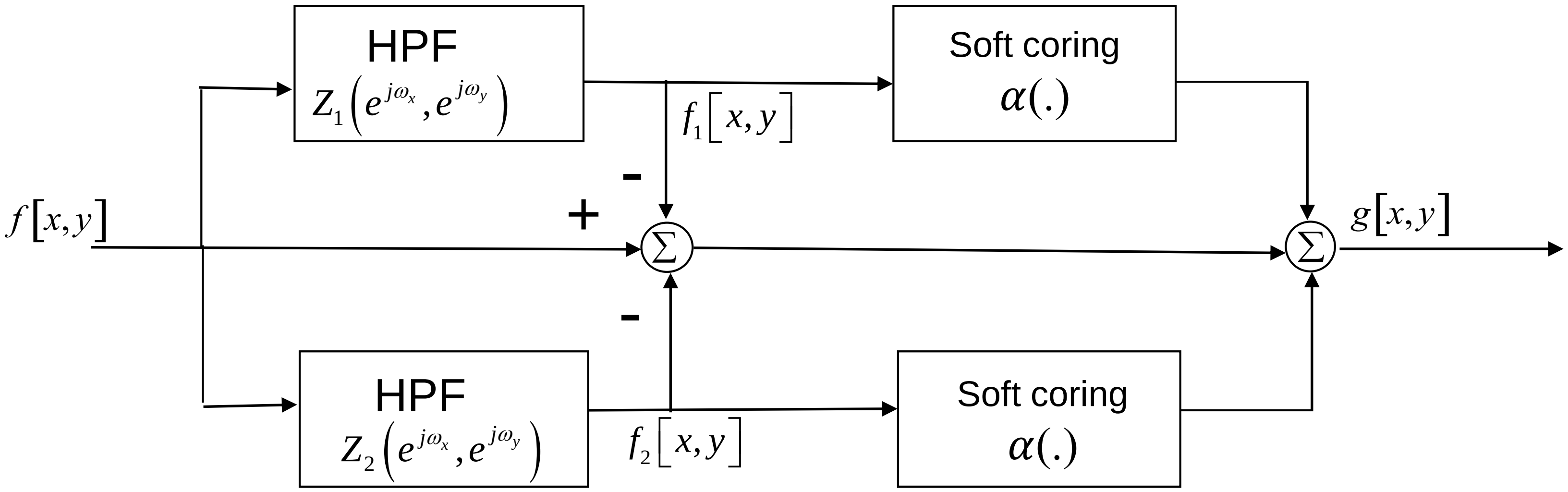
- Noise reduction: smooth the image, lowpass filtering
- Deblurring: sharpen edges, highpass filtering
- How can both be achieved simultaneously?
- Key insight: large amplitude of highpass filtered image indicates presence of edge

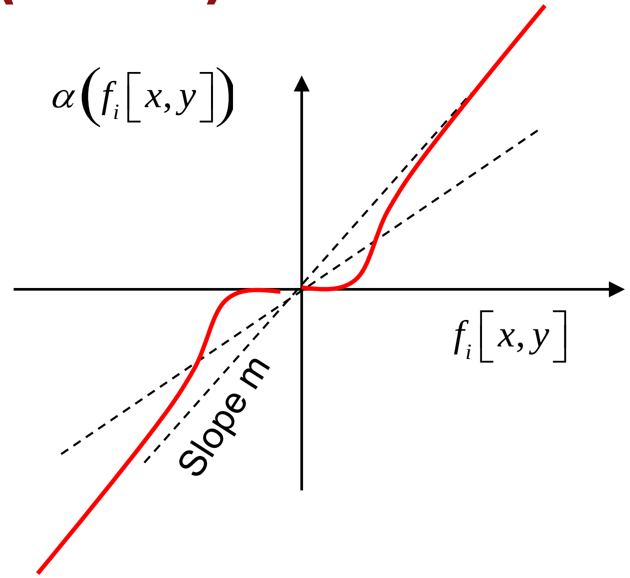



- Can be extended to multiple HPFs

# Nonlinear noise reduction/sharpening (cont.)

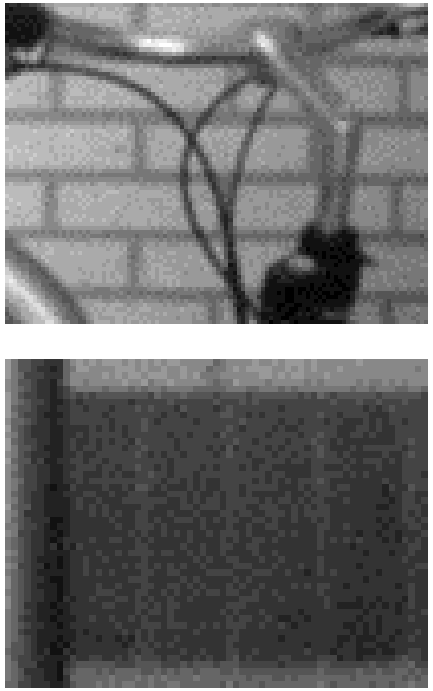


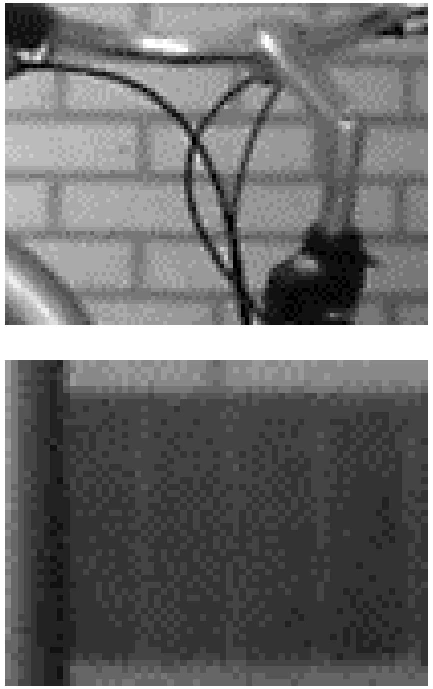




- Flat areas: $\alpha \approx 0 \;\rightarrow\; H\left(e^{j\omega_x},e^{j\omega_y}\right) \approx 1 - Z_1\left(e^{j\omega_x},e^{j\omega_y}\right) - Z_2\left(e^{j\omega_x},e^{j\omega_y}\right)$
- $f_1$ small; $f_2$ large: $H\left(e^{j\omega_x},e^{j\omega_y}\right) \approx 1 - Z_1\left(e^{j\omega_x},e^{j\omega_y}\right) + (m-1) Z_2\left(e^{j\omega_x},e^{j\omega_y}\right)$
- $f_1$ large; $f_2$ small: $H\left(e^{j\omega_x},e^{j\omega_y}\right) \approx 1 + (m-1) Z_1\left(e^{j\omega_x},e^{j\omega_y}\right) - Z_2\left(e^{j\omega_x},e^{j\omega_y}\right)$
- Both large: $H\left(e^{j\omega_x},e^{j\omega_y}\right) \approx 1 + (m-1)\left[Z_1\left(e^{j\omega_x},e^{j\omega_y}\right) + Z_2\left(e^{j\omega_x},e^{j\omega_y}\right)\right]$

# Nonlinear noise reduction/sharpening example

blurred, noisy image



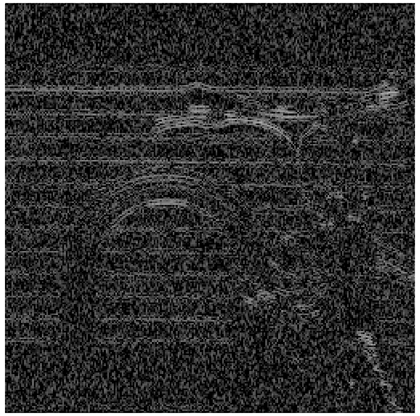


noise-reduced
and sharpened

# Highpass filtered images

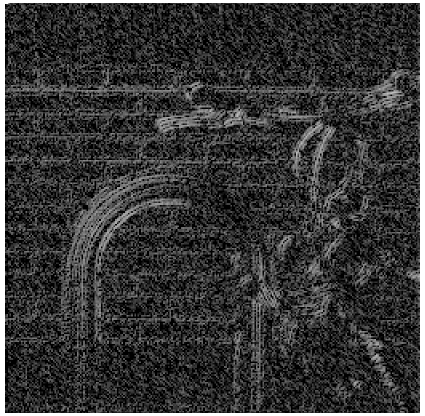
log magnitude of image filtered with

$$\begin{pmatrix} 0 & 0 & 0 \\ -0.5 & [1] & -0.5 \\ 0 & 0 & 0 \end{pmatrix}$$



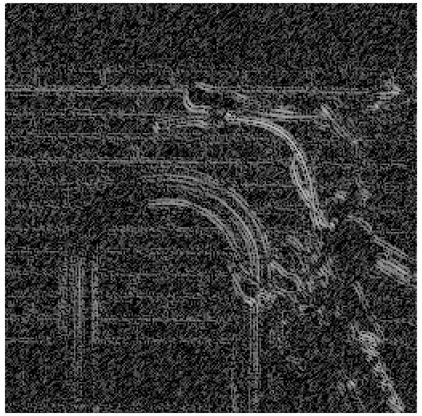
log magnitude of image filtered with

$$\begin{pmatrix} 0 & -0.5 & 0 \\ 0 & [1] & 0 \\ 0 & -0.5 & 0 \end{pmatrix}$$



log magnitude of image filtered with

$$\begin{pmatrix} -0.5 & 0 & 0 \\ 0 & [1] & 0 \\ 0 & 0 & -0.5 \end{pmatrix}$$

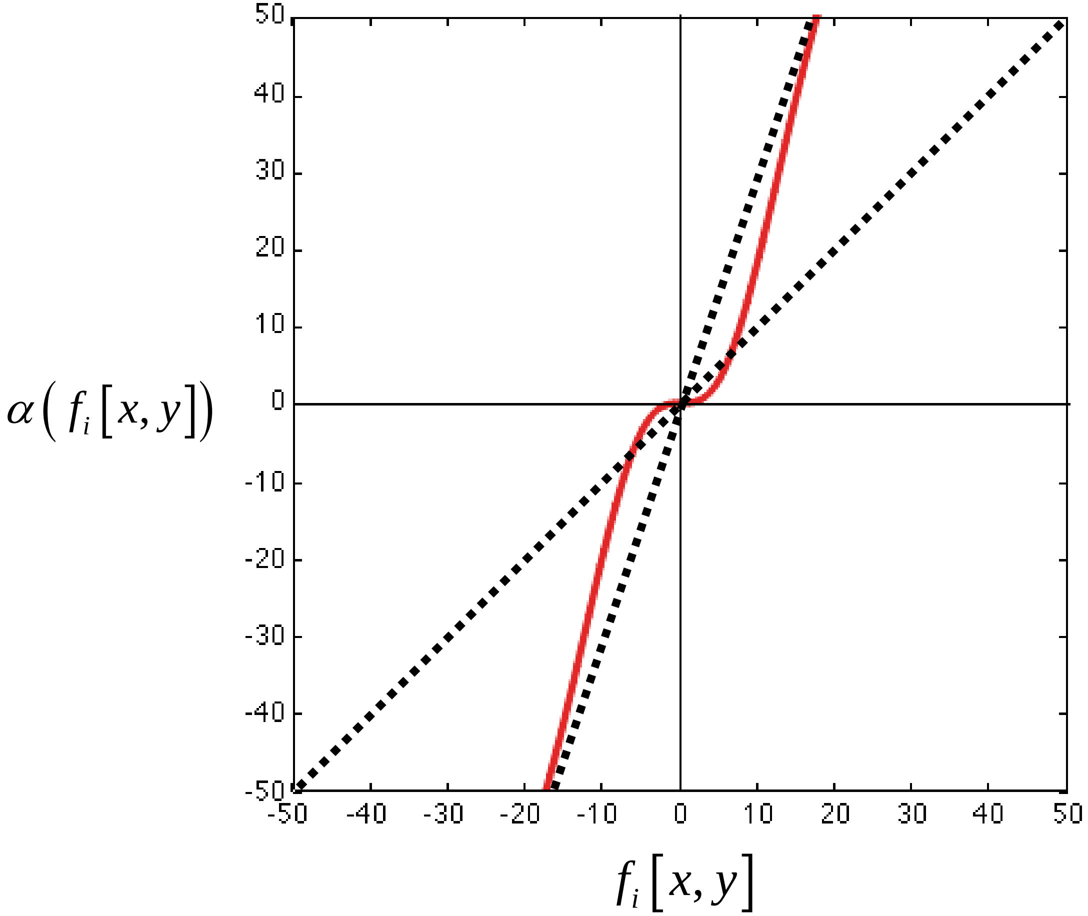


log magnitude of image filtered with

$$\begin{pmatrix} 0 & 0 & -0.5 \\ 0 & [1] & 0 \\ -0.5 & 0 & 0 \end{pmatrix}$$

# Soft coring function

$$\alpha\left(f_i[x,y]\right) = m \cdot f_i[x,y] \cdot \left(1 - e^{-\left|\frac{f_i[x,y]}{\tau}\right|^{\gamma}}\right)$$



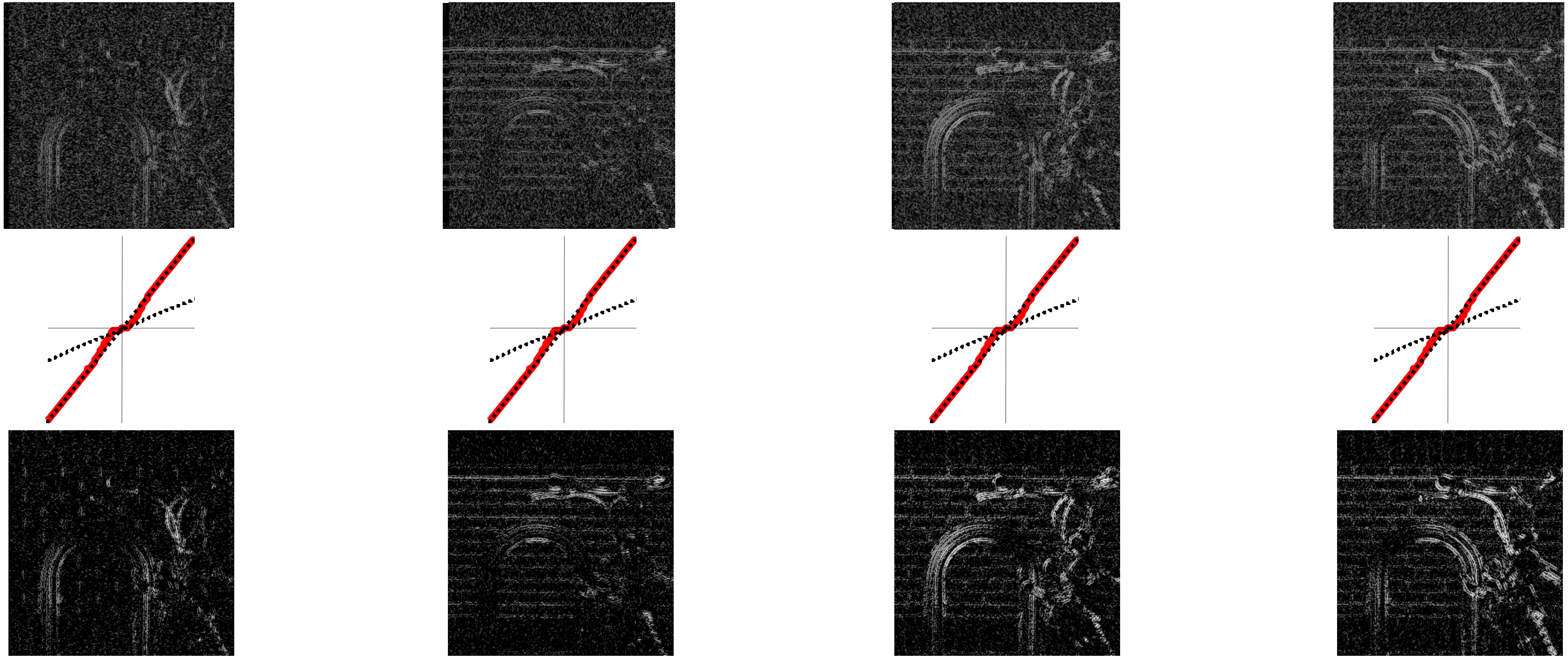

Example:

$m = 3$

$\gamma = 2$

$\tau = 10$

# Soft coring of highpass filtered images



# Linear vs. nonlinear noise reduction/sharpening

Noise reduction by lowpass filter (linear)

Sharpening by highpass filter (linear)

Combined noise reduction and sharpening (nonlinear)