PROJECTION RECONSTRUCTION

PARALLEL BEAM

FAU BEAM

BACK PROJECTION

REBINNING
Projection Reconstruction Problem

Given an object \( f(x,y) \) we collect projections.

How do we reconstruct \( f(x,y) \) from projections?

Many special cases

Parallel beam

Fan beam

General
Many modalities

X-ray  PET  SPECT  MRI  Optical  Ultrasound

Reconstruction Approaches

Parallel Beam - Central Slice Theorem, Backprojection
Convert to Parallel Beam
Backproject according to geometry
**Parallel Projection Data**

The projection data at an angle $\theta$ is

$$g(r, \theta) = \sum_{n} \sum_{m} f(x, y) \delta(x \cos \theta + y \sin \theta - r) \, dx \, dy$$

For X-ray CT

$$f(x, y) = m(x, y; E)$$

**Attenuation at Effective Energy $E$**

$$a(r, \theta) = -\log \left( \frac{I_0}{I} \right)$$

**Negative Log of Attenuation Intensity over Source Intensity**

How do we find $f(x, y)$ from $g(r, \theta)$?
Take 1D Fourier transform of projection:

\[
\hat{f}_0 \{ g(r, \theta) \} = \int_{-\infty}^{\infty} g(r, \theta) e^{-i2\pi kr} dr
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - r) dy dx \right] e^{-i2\pi kr} dr
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - r) e^{-i2\pi kr} dr dy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi (xk_\theta + yk_\psi)} dx dy
\]

\[
F(k_x, k_y) \bigg|_{k_x = k_r \cos \theta} \bigg|_{k_y = k_r \sin \theta}
\]

\[
= F(k_r \cos \theta, k_r \sin \theta)
\]

Projection at an angle \( \theta \) corresponds to a diameter in spatial frequency.
Repeat at many angles

Covers disc in spatial frequency

Fourier Reconstruction

Transform Projections
Gridding Interpolation
Inverse Transform
Density Correction

Previously, we showed that we can analytically compute the density

\[ N \] projections, e.g.

**Central Disk**

\[ \frac{N}{\text{samples}} \]

\[ (\frac{\alpha k r^2}{2}) \frac{\pi}{\text{area}} \]

**Ring**

\[ \frac{2N}{\text{samples}} \]

\[ (\alpha k r (n+\frac{1}{2}))^2 \pi - (\alpha k r (n-\frac{1}{2}))^2 \pi \]

**Area per sample** is \( a_n \):

\[ a_0 = \frac{(\alpha k r)^2 \pi}{N} \]

\[ H(k) = (a_0, a_1, \ldots, a_n) = \frac{(\alpha k r)^2}{N} \left( \frac{1}{2}, 1, 2, \ldots, n \right) \]

\[ a_n = \frac{(\alpha k r)^2 \pi}{N} \]

"RHO" Filter

Important that D.C. not be zero.

Other derivations give slightly different answers.
SUMMARY OF FOURIER DOMAIN APPROACH

1) Transform projections
2) Density correct (multiply by H(k))
3) Add to grid
4) Inverse transform

IN THE IMAGE DOMAIN THIS CORRESPONDS TO

1) Nothing
2) Convolve with h(r) = \mathcal{F}^{-1}\{H(k)\}
3) Spread filtered projection across image space
   "Backprojection"
4) Nothing

This is called

Filtered Backprojection
Convolution Backprojection

Very common
**IMPLEMENTATION**

**INITIAL DATA** is in RAdon SPACE $g(r, \theta)$

**SINOGRAM**: EACH POINT TRACES OUT A SINUSOID

**CONVOLVE** EACH PROJECTION WITH

$$h(n) = \mathcal{F}_{1D}\{H(kr)\}$$

**Often $H(kr)$ is windowed** to reduce ringing.

RESULT IS A **FILTERED SINOCGRAM**.
Two Approaches

1) Push ray through matrix, adding to pixels it crosses

2) Take the location of each pixel, and interpolate into projection data
FAN BEAM RECONSTRUCTION

MOST CT SYSTEMS USE A FAN BEAM GEOMETRY

TWO BASIC TYPES

EQUAL SPACED DETECTORS

\[ \theta \]

EQUAL ANGLE DETECTORS

\[ \theta \]

CIRCULAR ARC

WE WILL FOCUS ON THE EQUAL ANGLE CASE FOR NOW
**First Approach**

Convert the problem to a parallel beam problem rebinning.

**Basic Rebinning Idea**

Assume beams spaced by $\Delta \theta$, and that we rotate by $\Delta \theta$ steps.

\[ \theta = 0 \]

After increasing $\theta$ by $\Delta \theta$

\[ \theta = \Delta \theta \]
\[ \theta = 2 \Delta \theta \]

We can collect all these parallel beams into one parallel projection.

The edge rays are:

\[ \theta = -\gamma_m \]

\[ \theta = +\gamma_m \]

One projection requires \( 2 \gamma_m \), where the fan beam goes from \( -\gamma_m \) to \( +\gamma_m \).

Total angle required for reconstruction is \( \pi + 2 \gamma \).

Total number of projections:

\[ N = \frac{\pi + 2 \gamma_m}{\Delta \theta} \]
PRACTICAL ISSUES

Generally $\Delta \theta$ and $\Delta \phi$ are not the same

Although the rays are parallel after rebinning, they are not evenly spaced.

These are both interpolation problems we will talk about next time.