EE 369C LECTURE 2

1 D SAMPLING AND RECONSTRUCTION

FOURIER TRANSFORMS AND NOTATION

UNIFORM SAMPLING OF BANDLIMITED SIGNALS

OVERSAMPLING

NON-UNIFORM SAMPLING AND RECONSTRUCTION

SAMPLING ERRORS

EFFECTS OF SIMPLE INTERPOLATORS

1D FOURIER TRANSFORMS

$$S(k_x) = \int_{-\infty}^{\infty} s(x) e^{-j2\pi k_x x} dx \qquad \text{ANALYSIS}$$

$$s(x) = \int_{-\infty}^{\infty} S(k_x) e^{+j2\pi k_x x} dk_x \qquad \text{SYNTHESIS}$$

NOTATION

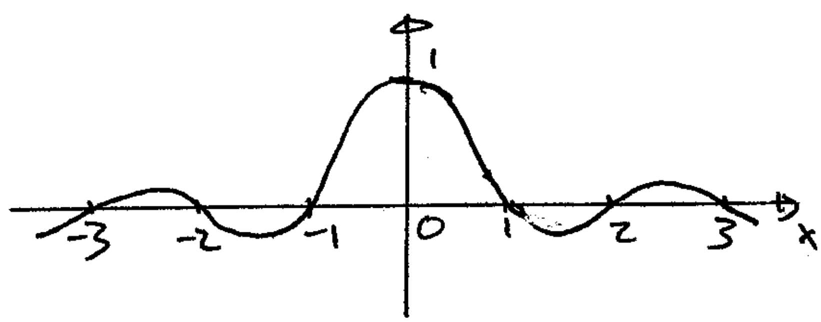
$$s(x) \Leftrightarrow S(k_x)$$

$$\mathcal{F}\{s(x)\} = S(k_x)$$

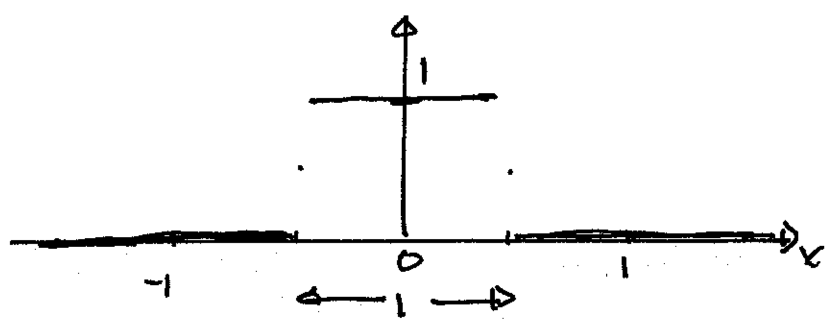
$$\mathcal{F}^{-1}\{S(k_x)\} = s(x)$$

FUNCTION DEFINITIONS

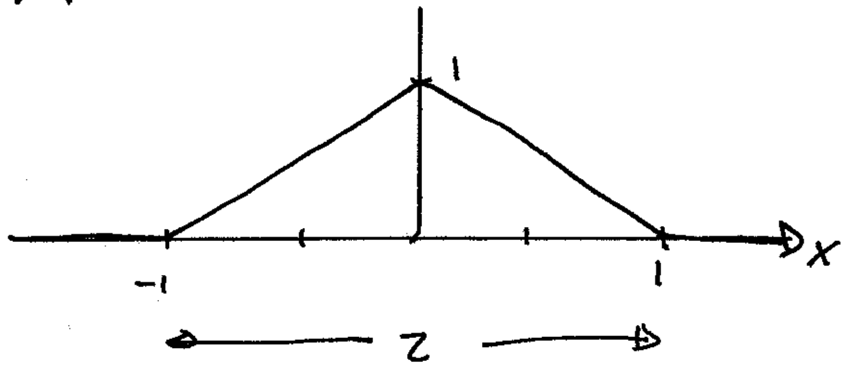
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}$$



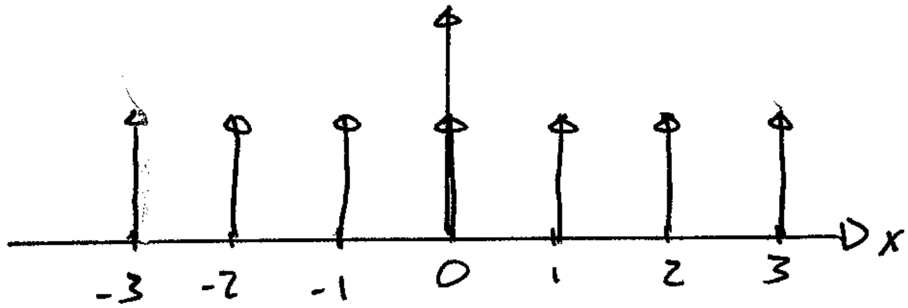
$$\sqcap(x) = \mathrm{rect}(x)$$



$\Lambda(x) = \text{tri}(x)$



$\text{Ш}(x) = \text{comb}(x) = \sum_n \delta(x-n)$



FOURIER TRANSFORM PAIRS

$\text{rect}(x) \Leftrightarrow \text{sinc}(k_x)$

$\text{sinc}(x) \Leftrightarrow \text{rect}(k_x)$

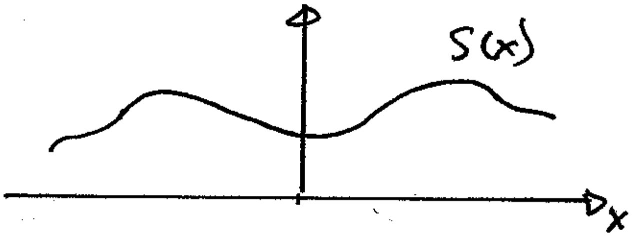
$\text{tri}(x) \Leftrightarrow \text{sinc}^2(k_x)$

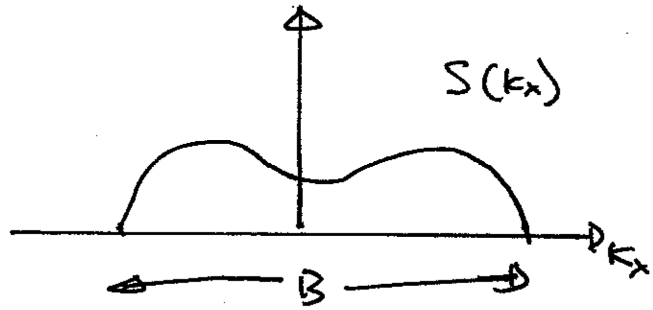
$\text{sinc}^2(x) \Leftrightarrow \text{tri}(k_x)$

$\text{Ш}(x) \Leftrightarrow \text{Ш}(k_x)$

UNIFORM SAMPLING

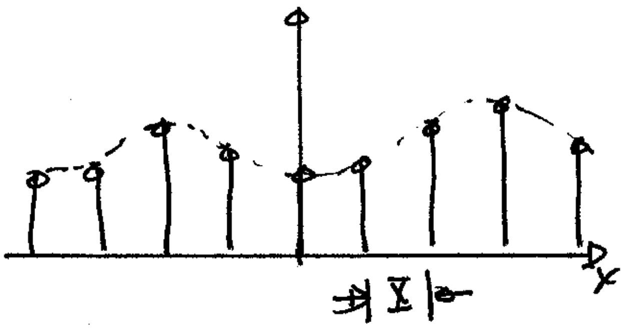
BANDLIMITED SIGNAL $s(x)$



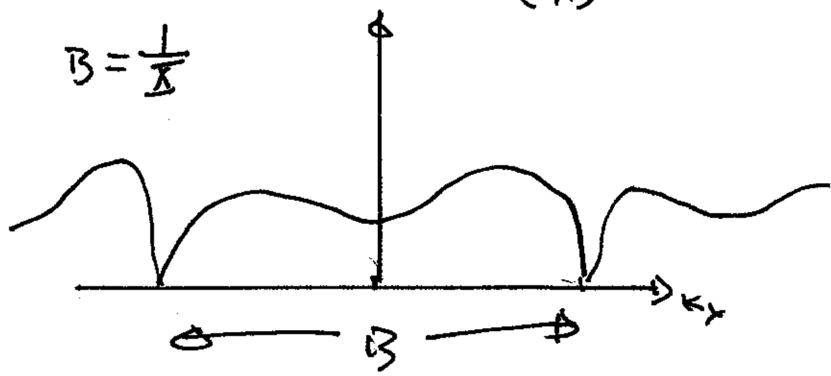


SAMPLED SIGNAL $s(x)\left(\frac{1}{X}\text{Ш}\left(\frac{x}{X}\right)\right)$

$S(k_x) * \text{Ш}(k_x X)$



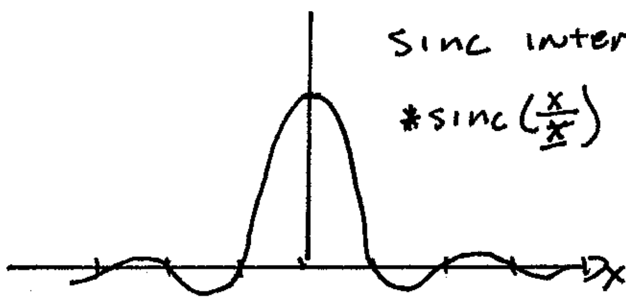
$B = \frac{1}{X}$



RECONSTRUCTION

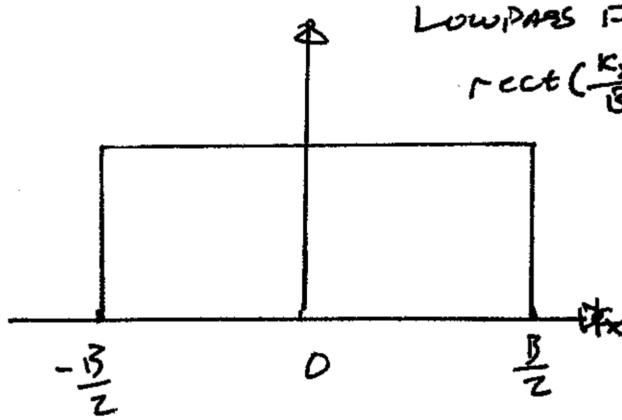
sinc interpolation

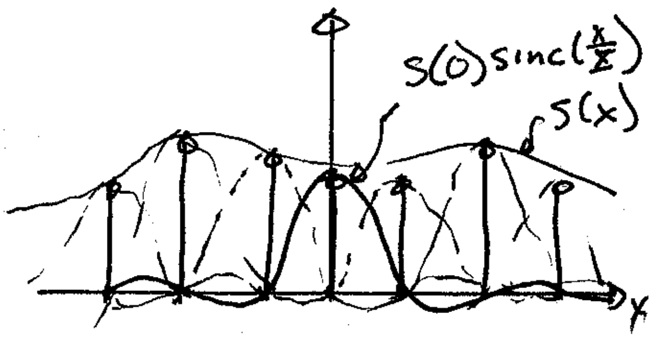
$* \operatorname{sinc}\left(\frac{x}{X}\right)$

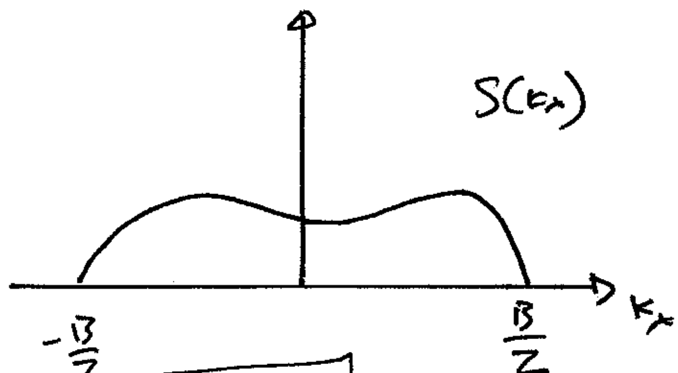


LOWPASS FILTER

$\operatorname{rect}\left(\frac{k_x}{B}\right)$







$$\boxed{s(x) = \sum_m s(mX)\operatorname{sinc}\left(\frac{x - mX}{X}\right)}$$

POSITIVE FEATURES

BEAUTIFUL RESULT

WHITAKER - SHANNON - KOTELNIKOV THEOREM

NICE INSIGHT

$\mathrm{sinc}\left(\frac{x}{X}\right)$ IS 1 AT ZERO, ZERO AT mX, $m \neq 0$

GUARANTEED THAT RECONSTRUCTED SIGNAL $s_r(x)$ IS

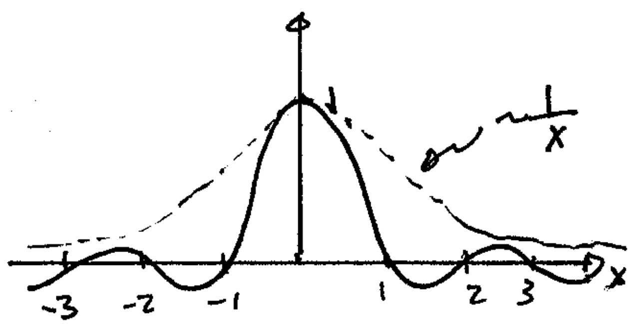
$s_r(x) = s(x)$ AT $x = nX$

$s_r(x)$ IS BANDLIMITED, WITH $BW = B$

ONLY <u>ONE</u> RECONSTRUCTION FILTER

NEGATIVE FEATURES

$\mathrm{sinc}(x)$ DECAYS SLOWLY, AS $1/x$



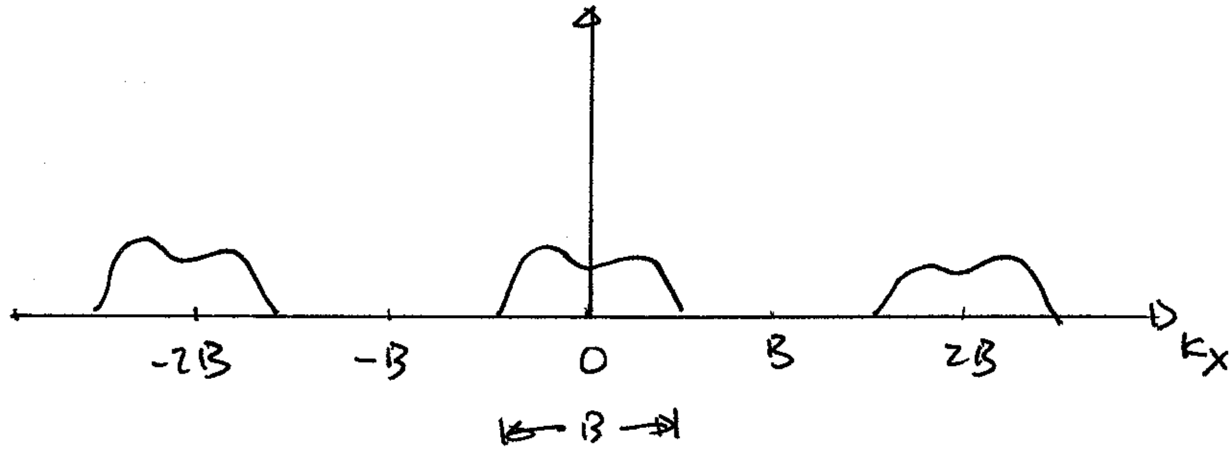
$$\left| \frac{\sin(\pi x)}{\pi x} \right| \leq \frac{1}{|\pi x|} \sim \frac{1}{x}$$

HIGH FIDELITY RECONSTRUCTIONS REQUIRE A LONG INTERPOLATION FILTER.

OVERSAMPLING

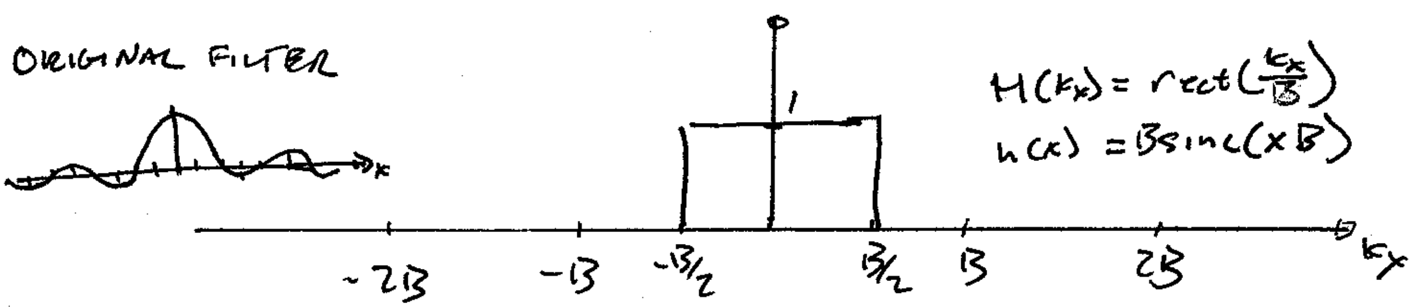
SITUATION IMPROVES IF WE SAMPLE FASTER THAN B

FOR EXAMPLE IF $\frac{1}{X} = 2B$



NOW THERE ARE LOTS OF FILTERS THAT WORK

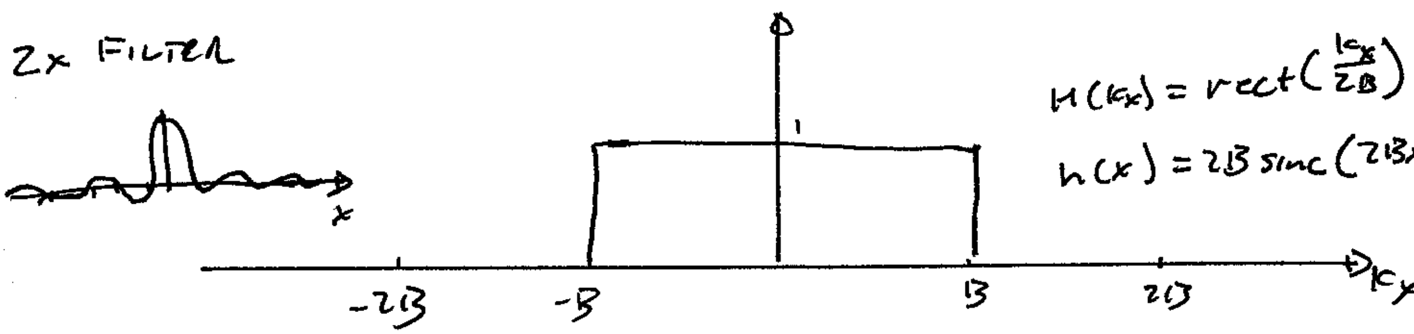
ORIGINAL FILTER



$H(k_x) = rect\left(\frac{k_x}{B}\right)$

$h(x) = B\,sinc(xB)$

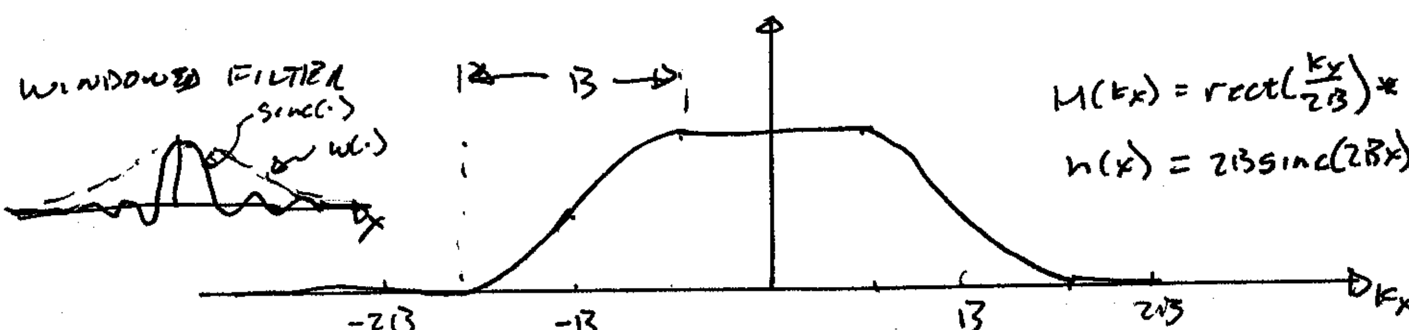
2x FILTER



$H(k_x) = rect\left(\frac{k_x}{2B}\right)$

$h(x) = 2B\,sinc(2Bx)$

WINDOWED FILTER



$H(k_x) = rect\left(\frac{k_x}{2B}\right) * W\left(\frac{k_x}{B/2}\right)$

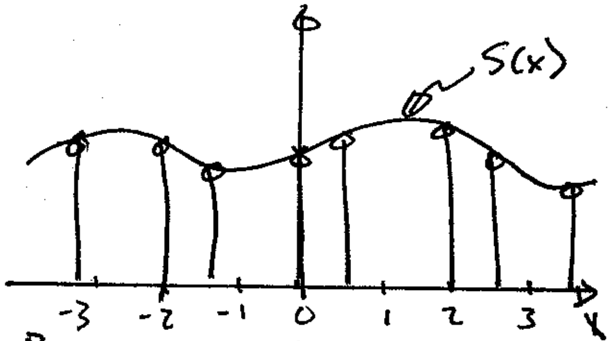
$h(x) = 2B\,sinc(2Bx)\,w\left(\frac{B}{2}x\right)$

<u>MUCH</u> SHORTER INTERPOLATION FILTER

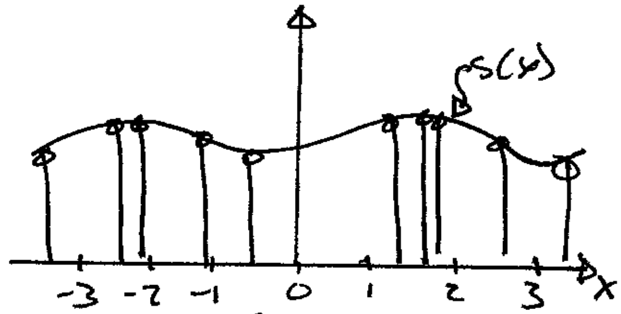
L2

NON-UNIFORM SAMPLING

OFTEN SAMPLES ARE NOT UNIFORMLY SPACED



BUNCHED SAMPLING



RANDOM SAMPLING

QUESTIONS

CAN WE STILL RECONSTRUCT $s(x)$?

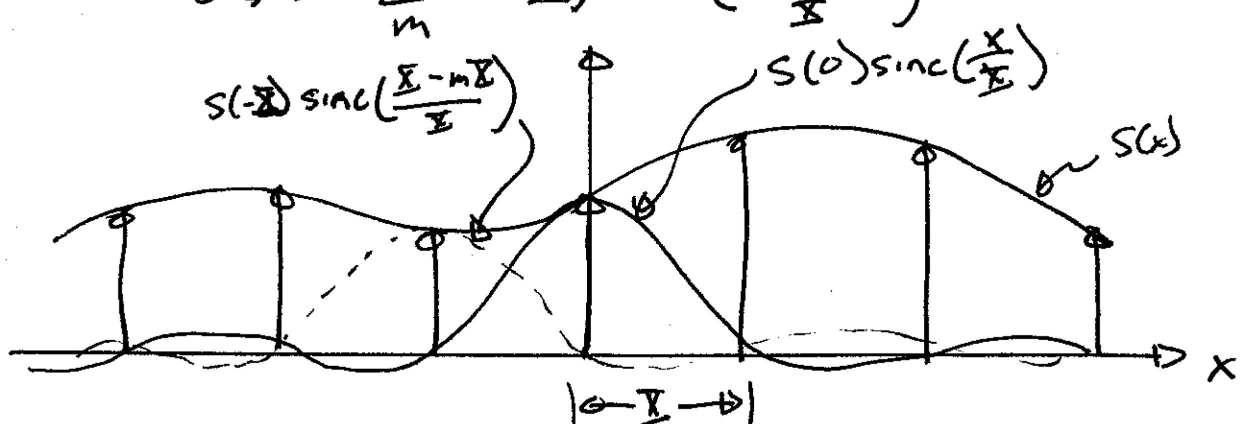
HOW MANY SAMPLES DO WE NEED?

HOW DO WE DO THE RECONSTRUCTION?

BASIC IDEA

SINCE $s(x)$ IS BANDLIMITED, WE CAN WRITE IT AS

$$s(x) = \sum_m s(m\Sigma)\,\mathrm{sinc}\left(\frac{x - m\Sigma}{\Sigma}\right)$$



IF WE CAN FIND $s(m\Sigma)$, WE CAN RECOVER $s(x)$

LET

$$x_m = m\,\Delta x \qquad \text{UNIFORM SAMPLES}$$

$$x_n \qquad \text{NON-UNIFORM SAMPLES}$$

IF WE KNEW $s(x_m)$, WE COULD SOLVE FOR $s(x_n)$

$$s(x_n) = \sum_m s(x_m)\,\mathrm{sinc}\left(\frac{x_n - x_m}{\Delta x}\right)$$

THIS IS AN INFINITE SUM. IN PRACTICE THIS WILL BE LIMITED TO $m < M$, $n < N$. WE CAN THEN WRITE THIS AS A MATRIX EQUATION

$$\underset{\substack{N \times 1 \\ \text{NON-UNIFORM} \\ \text{SAMPLES}}}{\begin{bmatrix} \vdots \\ s(x_n) \\ \vdots \end{bmatrix}} = \underset{\substack{N \times m \\ \text{INTERPOLATION} \\ \text{MATRIX}}}{\begin{bmatrix} & & \\ & \mathrm{sinc}\left(\frac{x_n - x_m}{\Delta x}\right) & \\ & & \end{bmatrix}} \underset{\substack{m \times 1 \\ \text{UNIFORM} \\ \text{SAMPLES}}}{\begin{bmatrix} \vdots \\ s(x_m) \\ \vdots \end{bmatrix}}$$

$$\underset{\text{KNOWN}}{\underline{s}_n} = \underset{\text{COMPUTED}}{\underline{\underline{E}}} \cdot \underset{\text{UNKNOWN}}{\underline{s}_u}$$

SOLUTION

IF $N > M$, AND THE x_u ARE UNIQUE, THIS CAN BE SOLVED BY A PSEUDOINVERSE

$$\underline{s}_u = \left(E^* E\right)^{-1} E^* \underline{s}_n$$

IN MATLAB, USE LEFT MATRIX DIVIDE

$$su = E \backslash sn$$

LEAST SQUARES SOLUTION

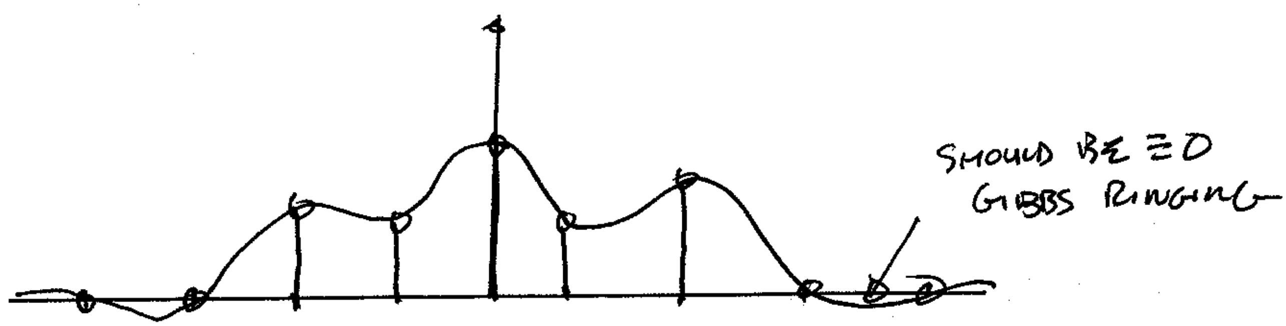
GIVEN $\underline{s}_u$, WE CAN RECONSTRUCT $s(x)$ ANYWHERE.

PRACTICAL ISSUE

SINC INTERPOLATOR DECAYS SLOWLY

WE ONLY HAVE FINITE INTERVALS OF SAMPLES

RESULT IS EDGE EFFECTS



MINIMIZE BY OVERSAMPLING

SIMPLE INTERPOLATORS

SINC INTERPOLATION IS EXPENSIVE

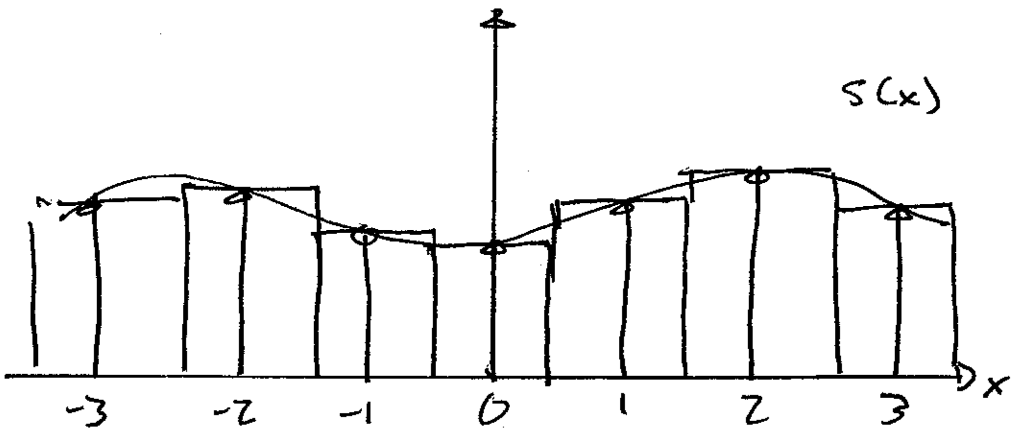
HOW WELL DO SIMPLE INTERPOLATORS DO?

ZERO ORDER HOLD

LINEAR INTERPOLATION

WE CAN DESCRIBE THESE AS CONVOLUTIONS

ZOH



SAMPLED SIGNAL

$$s(x)\,\frac{1}{X}\,\text{Ш}\left(\frac{x}{X}\right)$$

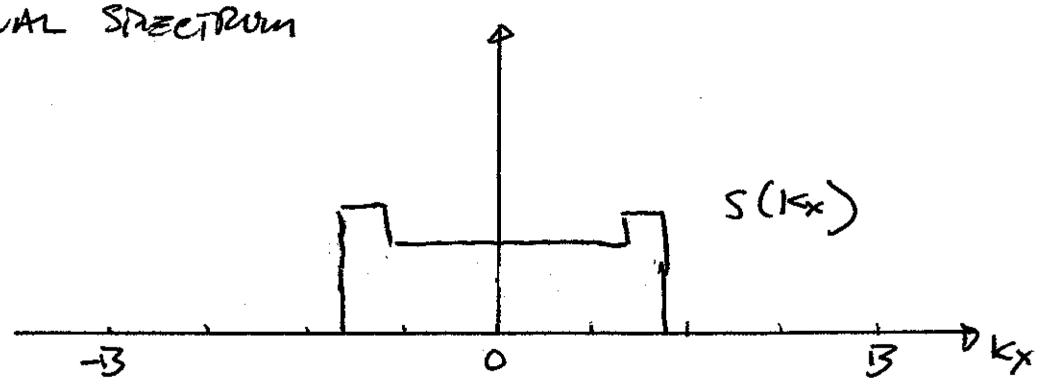
RECONSTRUCTED SIGNAL

$$\left(s(x)\,\frac{1}{X}\,\text{Ш}\left(\frac{x}{X}\right)\right) * \mathrm{rect}\left(\frac{x}{X}\right)$$

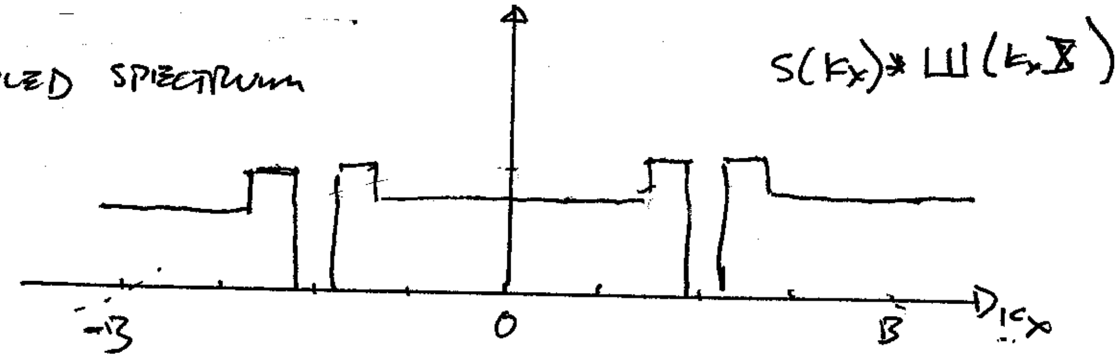
FREQUENCY DOMAIN

$$\left(S(k_x) * \text{Ш}(k_x X)\right) \cdot X\,\mathrm{sinc}(k_x X)$$

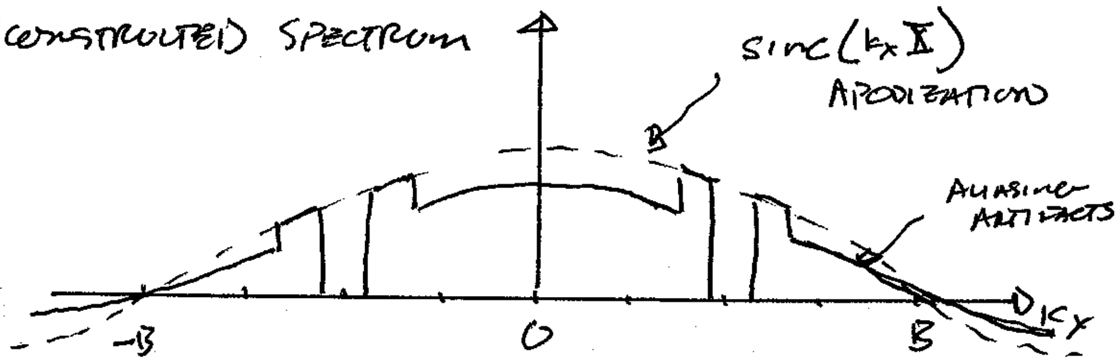
ORIGINAL SPECTRUM



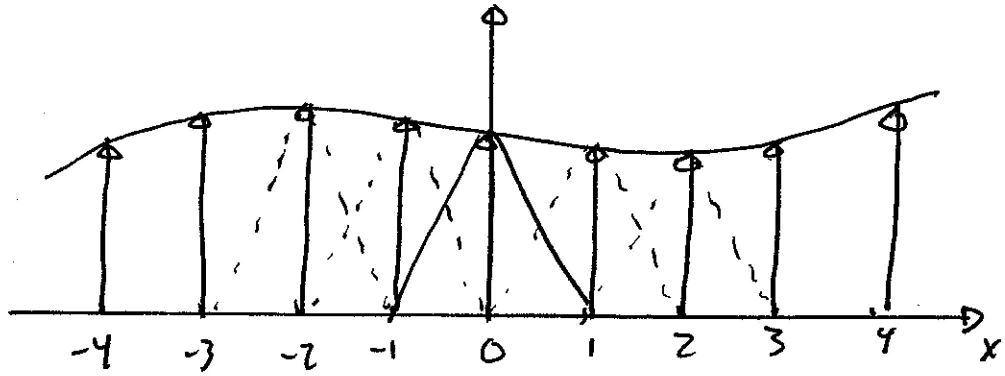
SAMPLED SPECTRUM



RECONSTRUCTED SPECTRUM



LINEAR INTERPOLATION

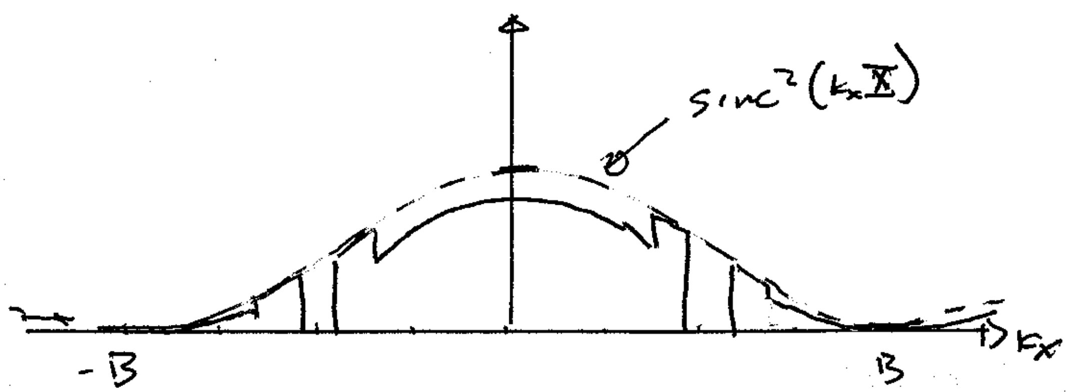


CONVOLUTION WITH $\mathrm{TRI}\left(\frac{x}{X}\right)$

$$\left(s(x)\frac{1}{X}\text{Ш}\left(\frac{x}{X}\right)\right) * \mathrm{TRI}\left(\frac{x}{X}\right)$$

RECONSTRUCTED SPECTRUM

$$\left(S(k_x) * \text{Ш}(k_x X)\right)\cdot X\,\mathrm{sinc}^2(k_x X)$$



MORE APODIZATION

REDUCED ALIASING ARTIFACTS

APPROXIMATION TO A SINC