GRIDDING KERNEL DESIGN AND OVERSAMPLING

ASSIGNMENT 2
READ BEATTY PAPER

LAST TIME
GRIDDING
DENSITY COMPENSATION

THIS TIME
KERNEL DESIGN
OVERSAMPLING RATIO
KERNEL SAMPLING
**Problem with 1x Grid**

**2x Grid**

No Transition Band

Aliased Signal Becomes Regularized Signal

Sample twice as finely in $k$-space.

Spatial replicas twice as far out.

Many kernels work.

What should we choose?
RECT KERNEL

\[ C(k_x) = \text{rect}(\frac{k_x}{2\Delta k}) \]

EASY TO COMPUTE

NEAREST NEIGHBOR, ONLY HITS ONE GRID POINT

IN IMAGE SPACE

\[ C(k) = \frac{1}{F(V_k)} \sin c\left(\frac{k}{F(V_k)}\right) \]

WE ARE GOING TO DE-APOXIZE, SO WHAT WE CARE ABOUT IS RATIO OF MAINLOBE TO SIDELOBE

VOLUME AT BAND EDGE

MAIN LOBE

\[ \sin c\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} = \frac{\pi}{4} \approx 0.64 \]

SIDE LOBES

\[ |\sin c\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)| = \left|\frac{\sin\left(\frac{3\pi}{8}\right)}{\frac{3\pi}{8}}\right| = \frac{3\pi}{8} \approx 0.71 \]

RATIO IS \( \frac{1}{3} \) NOT GOOD

HIGHER OVER-Sampling CAN BE AS GOOD AS YOU LIKE!
TRIANGLE KERNEL

\[ C(k_x) = \text{tanh} \left( \frac{k_x}{\Delta k_x} \right) \]

EASY TO COMPUTE (DISTANCE)
LINEAR INTERPOLATION IN K-SPACE
DATA HITS FOUR GRID POINTS, BILINEAR INTERPOLATION IN IMAGE SPACE

\[ c(x) = \frac{1}{ \text{FOV}_x } \sin^2 \left( \frac{k}{ \text{FOV}_x } \right) \]

MAIN LOBE AT BAND EDGE
\[ \sin^2 \left( \frac{k}{\pi} \right) = \left( \frac{2}{\pi} \right)^2 \approx 0.41 \]

SIDE LOBE AT BAND EDGE
\[ \sin^2 \left( \frac{2k}{3\pi} \right) = \left( \frac{2}{3\pi} \right)^2 \approx 0.045 \]

RATIO IS \[ \frac{1}{9} \]
THIS IS OFTEN GOOD ENOUGH!
WE CAN DO MUCH BETTER
Window Function Kernels

Many smooth lowpass functions

Window Functions

Here, Kaiser-Bessel window

\[ f(k) = \frac{1}{k_0} I_0 \left( \beta \sqrt{1 - \left( \frac{k}{k_0} \right)^2} \right) \]

Where

\[ \beta - \text{shape parameter} \]

\[ k_0 - \text{width in spatial frequency} \]

\[ I_0(\cdot) - \text{zero order modified Bessel function of the first kind (built into MATLAB)} \]

How do we choose \( \beta, k_0 \)?

For 2x case, analyzed by Jackson, et al.

Given \( k_0 \), provides \( \beta \) to minimize aliasing.
EXAMPLE

4 SAMPLE 1-3 KERNEL

From Jackson, \( \beta = 9 \)

Same computation as \( \text{tri}(\cdot) \)

Main lobe at band edge

\( \approx 0.5 \)

1st side lobe at band edge

\( \approx 0.5 \times 10^{-3} \)

Ratio is \( \frac{1}{10^3} \)

Much better than required for \( \mu_{1/2} \).
REDUCED OVERSAMPLING RATIO

WITH 2x GRID, A 4 SAMPLE KERNEL GIVES AN ALMOST PERFECT RECON.

HOW MUCH CAN I REDUCE THE OVERSAMPLING, AND STILL HIT A SPECIFIC ALIASING LEVEL?

A LOT

NEEDS FPTW TO MAKE SENSE

OPTIMUM KB KERNEL

GIVEN A KERNEL WIDTH \( W \) IN OVERSAMPLED GRID UNITS

\[
\frac{W}{4}
\]

AND AN OVERSAMPLING FACTOR \( L \), FIND \( \beta \) TO MINIMIZE ALIASING
**Solution 1** (Water, Iswurm 1999)

![Diagram of solution 1]

Put zero of \( c(x) \) at band edge:

\[
\beta = \pi \sqrt{\frac{u}{\alpha} (\alpha - \frac{1}{2})^2 - 0.8}
\]

**Solution 2** (Bizzati)

![Diagram of solution 2]

Put zero inside FOV:

\[
\beta = \pi \sqrt{\frac{u}{\alpha} (\alpha - \frac{1}{2})^2 - 0.8}
\]

Much better as \( \alpha \to 1 \)
W vs \( \beta \)

\( \beta \) is a strong function of \( \omega = w \)

All kernels similar with respect to \( \Delta x \)

As \( W \) increases, \( C(\beta, \omega) \) gets smoother

\( W \) and \( \omega \) vs Aliasing

For a given aliasing amplitude, what \( W \) and \( \omega \) should I choose?

Fig 3 in Beatty Paper

\( 0.1 \times W \) ALIASING

\( \omega = 1.125 \), \( W = 8 \)

\( \omega = 1.25 \), \( W = 6 \)

\( \omega = 2 \), \( W = 4 \)
KERNEL SAMPLING

KERNEL IS RECOMPUTED MANY TIMES DOMINATES COMPUTATION

HOW CAN I PRECOMPUTE KERNEL

COMPUTE AT $S$ SAMPLES PER POINT

$N$ NEAREST NEIGHBOUR LOOK UP

$N$ LINEAR INTERPOLATION

SURPRISINGLY HUGE DIFFERENCE

$\text{NEAREST NEIGHBOUR}$

$$\max(\varepsilon_i) = \frac{0.91}{\alpha S}$$

$\text{LINEAR INTERPOLATION}$

$$\max(\varepsilon_i) = \frac{0.37}{(\alpha S)^2}$$

$\alpha$ - OVERSAMPLING RATIO

$S$ - KERNEL OVERSAMPLING

$\varepsilon_i$ - AVERAGE ERROR OVER $120V$

EXAMPLE

$S = 10^4$, ASSUME $\alpha = 1.25$, $W = 6$ (AVERAGE AMPLITUDE $0.1 \times 10^{-5}$)

$\text{NEAREST NEIGHBOUR}$

$S = 7280$ \hspace{1cm} $WS = (7280)(6) = 43,680$ SAMPLES

$\text{LINEAR}$

$S = 49$ \hspace{1cm} $WS = 6(49) = 294$ SAMPLES

IMPORTANT BECAUSE MEMORY MORE IMPORTANT THAN COMPUTATION OFTEN.