EE376A: Homework #2
Due on Thursday, January 29, 2015

You can hand in the homework either after class or deposit it, before 5 PM, in the EE376A drawer of the class file cabinet on the second floor of the Packard Building.

Some definitions that may be useful:

Definition 1: The conditional mutual information of random variables $X$ and $Y$ given $Z$ is defined by

$$I(X; Y | Z) = H(X | Z) - H(X | Y, Z)$$

$$= \sum_{x,y,z} P(x, y, z) \log \frac{P(x, y | z)}{P(x | z) P(y | z)}$$

Definition 2: A sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ converges to a random variable $X$ in probability if for any $\epsilon > 0$,

$$\lim_{n \to \infty} P\{|X_n - X| < \epsilon\} = 1$$

1. Data Processing Inequality.
   If $X, Y, Z$ form a markov triplet ($X - Y - Z$), show that:
   
   (a) $H(X | Y) = H(X | Y, Z)$ and $H(Z | Y) = H(Z | X, Y)$
   (b) $H(X | Y) \leq H(X | Z)$
   (c) $I(X; Y) \geq I(X; Z)$ and $I(Y; Z) \geq I(X; Z)$
   (d) $I(X; Z | Y) = 0$

2. Entropy and pairwise independence.
   Let $X, Y, Z$ be three binary Bernoulli($\frac{1}{2}$) random variables that are pairwise independent; that is, $I(X; Y) = I(X; Z) = I(Y; Z) = 0$.

   (a) Under this constraint, what is the minimum value for $H(X, Y, Z)$ (under all possible joint distributions of $(X, Y, Z)$ that are consistent with the above)?
   (b) Give an example achieving this minimum.
   (c) Now suppose that $X, Y, Z$ are three random variables each uniformly distributed over the alphabet $\{1, 2, \ldots, m\}$. Again, they are pairwise independent. What is the minimum value for $H(X, Y, Z)$?

3. Conditional entropy. Let $X_1, X_2, X_3$ be i.i.d. discrete random variables.
   Which is larger: $H(X_1 | X_1 + X_2 + X_3)$ or $H(X_1 + X_2 | X_1 + X_2 + X_3)$?
4. Inequalities.
Let $X$, $Y$ and $Z$ be joint random variables. Prove the following inequalities and find conditions for equality.

(a) $H(X,Y | Z) \geq H(X | Z)$.
(b) $I(X,Y ; Z) \geq I(X; Z)$.
(c) $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$.
(d) $I(X; Z | Y) \geq I(Z; Y | X) - I(Z; Y) + I(X; Z)$.

5. Conditional mutual information.
Consider a sequence of $n$ binary random variables $X_1, X_2, \ldots, X_n$. Each $n$-sequence with an even number of 1’s has probability $2^{-(n-1)}$ and each $n$-sequence with an odd number of 1’s has probability 0. Find the mutual informations

$I(X_1; X_2), I(X_2; X_3 | X_1), \ldots, I(X_{n-1}; X_n | X_1, \ldots, X_{n-2})$.

6. The value of a question.
Let $X \sim p(x)$, $x = 1, 2, \ldots, m$. We are given a set $S \subseteq \{1, 2, \ldots, m\}$. We ask whether $X \in S$ and receive the answer

$Y = \begin{cases} 1, & \text{if } X \in S \\ 0, & \text{if } X \notin S. \end{cases}$

Suppose $P\{X \in S\} = \alpha$. Find the decrease in uncertainty $H(X) - H(X|Y)$.

7. AEP
Let $X_i$ be iid $\sim p(x), x \in \{1, 2, \ldots, m\}$. For $\epsilon > 0$, let $\mu = EX$, and $H = -\sum p(x) \log p(x)$. Let $A_n = \{x^n \in \mathcal{X}^n : \left| -\frac{1}{n} \log p(x^n) - H \right| \leq \epsilon \}$ and $B_n = \{x^n \in \mathcal{X}^n : \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| \leq \epsilon \}$.

(a) Does $P\{X^n \in A_n\} \rightarrow 1$?
(b) Does $P\{X^n \in A_n \cap B_n\} \rightarrow 1$?
(c) Show $|A_n \cap B_n| \leq 2^{n(H+\epsilon)}$, for all $n$.
(d) Show $|A_n \cap B_n| \geq (\frac{1}{2})2^{n(H-\epsilon)}$, for $n$ sufficiently large.
8. Let $X_1, X_2, \ldots$ be i.i.d. $\sim X$ where $X \in \mathcal{X} = \{1, 2, \ldots, K\}$ and $p(x) > 0$ for all $x \in \mathcal{X}$.

(a) Find the limit (in probability) of
\[ (p(X_1, X_2, \ldots, X_n))^\frac{r}{n} \]
as $n \to \infty$. ($r$ is fixed.)

(b) Find
\[ E \left( \frac{1}{p(X_1, X_2, \ldots, X_n)} \right). \]

9. An AEP-like limit and the AEP.

(a) Let $X_1, X_2, \ldots$ be i.i.d. drawn according to probability mass function $p(x)$. Find the limit in probability as $n \to \infty$ of
\[ p(X_1, X_2, \ldots, X_n)^{\frac{1}{n}}. \]

(b) Let $X_1, X_2, \ldots$ be drawn i.i.d. according to the following distribution:
\[
X_i = \begin{cases} 
2, & \text{with probability } \frac{2}{5} \\
3, & \text{with probability } \frac{2}{5} \\
4, & \text{with probability } \frac{1}{5}
\end{cases}
\]
Find the limit (in probability) of the product
\[ (X_1X_2\cdots X_n)^{1/n}. \]

(c) Evaluate the limit of $p(X_1, X_2, \ldots, X_n)^{\frac{2}{n}}$ assuming the distribution in part (b).