1 Joint Source-Channel Coding

1.1 Recap from last lecture

The goal is to communicate the $U^N = (U_1, U_2, \ldots, U_N)$ through the memoryless channel given by $P_{Y|X}$ with small expected distortion, measured by $E[d(U^N, V^N)]$. Note that the $U_i$ are not necessarily bits. A schematic of this setup is shown below:

The rate of communication is then

$$\text{rate} = \frac{\text{N source symbols}}{\text{channel use}}$$

We also allow an expected distortion $E[d(U^N, V^N)]$.

A rate-distortion pair $(\rho, D)$ is achievable if $\forall \epsilon > 0 \exists$ scheme with $\frac{N}{n} \geq D + \epsilon$ and $E[d(U^N, V^N)] \leq D + \epsilon$

Last lecture, we proved the “Source-Channel Separation Theorem”:

$(\rho, D)$ is achievable if and only if $\rho R(D) \leq C$

We can achieve points on the curve above by first thinking about representing the data as bits in an efficient manner (compression) with rate $R(D)$ and then transmitting these bits losslessly across the channel with rate $C$. Note that the distortion without sending anything over the channel is $D_{max}$.

Example 1. Binary Source and Binary Channel

Source: $U \sim \text{Ber}(p), \quad 0 \leq p \leq 1/2$
Channel: BSC(q), 0 ≤ q ≤ 1/2
Distortion: Hamming

Recall that for $U \sim Ber(p)$, the rate distortion function is $R(D) = h_2(p) - h_2(D)$ and that a binary symmetric channel with crossover probability $q$ has capacity $C = 1 - h_2(q)$

So, we see that if we want distortion $\leq D$, then (for $D \leq p$) the maximum achievable rate is:

$$\rho = \frac{1 - h_2(q)}{h_2(p) - h_2(D)}$$

Note that the communication problem corresponds to $D = 0$.

In particular, if $p = 1/2$, then if we want distortion $\leq D$, the maximum rate we can transmit at is:

$$\rho = \frac{1 - h_2(q)}{1 - h_2(D)}$$

Consider the following scheme for rate=1:

Channel input: $X_i = U_i$
reconstruction: $V_i = Y_i$

The expected distortion is then $P(U_i \neq V_i) = P(X_i \neq Y_i) = q$

→ this scheme is optimal, since $\rho = (1 - h_2(q))/(1 - h_2(D = q)) = 1$.

In this particular case, it is possible to achieve the optimal rate using a scheme that individually encodes and transmits each symbol.

**Example 2.** Gaussian Source and Gaussian Channel
Source: $U \sim \mathcal{N}(1, \sigma^2)$

Channel: AWGN (Additive White Gaussian Noise Channel) with power constraint $P$

Channel model:

$Z \sim (0,1)$

$X \rightarrow + \rightarrow Y$

distortion: squared error

Recall that for $U \sim \mathcal{N}(1, \sigma^2)$, the rate distortion function is $R(D) = \frac{1}{2} \log(\frac{\sigma^2}{D})$ (for $0 \leq D \leq \sigma^2$) and that the AWGN channel with power constraint $P$ has capacity $C = \frac{1}{2} \log(1 + P)$

Then, for a given distortion $D \leq \sigma^2$, the maximum achievable rate is

$$\rho = \frac{\log(1 + P)}{\log(\sigma^2/D)}$$

Consider the following scheme at rate=1:

transmit: $X_i = \sqrt{\frac{P}{\sigma^2}} U_i$

receive: $Y_i = X_i + Z_i = \sqrt{\frac{P}{\sigma^2}} U_i + Z_i$

reconstruction: $V_i = \mathbb{E}[U_i | V_i]$ 

The distortion is squared error, so we know that reconstruction using the expected value is optimal. Thus, we take $V_i = \mathbb{E}[U_i | V_i]$. 

The expected distortion is then:
\[ \mathbb{E}[U_i|V_i] = \text{Var}(U_i|V_i) \]
\[ = \text{Var} \left( U_i \mid \sqrt{\frac{\sigma^2}{P} Y_i} \right) \]
\[ = \text{Var} \left( U_i \mid U_i + \sqrt{\frac{\sigma^2}{P} Z_i} \right) \]
\[ \overset{(a)}{=} \frac{\sigma^2(\sigma^2/P)}{\sigma^2 + \sigma^2/P} \]
\[ = \frac{\sigma^2}{P + 1} \]

where (a) follows from the fact that for \( X \sim \mathcal{N}(0, \sigma^2) \) independent from \( Y \sim \mathcal{N}(0, \sigma^2) \):

\[ \text{Var}(X|X+Y) = \frac{\sigma^2 \sigma^2}{\sigma^2 + \sigma^2} \]

Now, at rate = 1:
The optimal \( D \) satisfies

\[ \frac{\log(1 + P)}{\log(\sigma^2/D)} = 1 \]
\[ \rightarrow 1 + P = \frac{\sigma^2}{D} \]

So, in the specific case of rate = 1 we see that the simple scheme above is optimal, just as the simple scheme for the Binary Source and Channel was also optimal when rate = 1.