Homework Set #5 Solutions

1. Problem 6.6.

(a) The decoding scheme is equivalent to successive decoding for two DM-MAC 
$p(y_1|x_1, x_2)$ and $p(y_2|x_1, x_2)$ sharing the same inputs $X_1$ and $X_2$. Therefore the 
rate region is given by the set of all $(R_1, R_2)$ such that 
\[
R_1 \leq \min\{I(X_1; Y_1|X_2, Q), I(X_1; Y_2|Q)\},
\]
\[
R_2 \leq \min\{I(X_2; Y_1|Q), I(X_2; Y_2|X_1, Q)\},
\]
for some pmf $p(q)p(x_1|q)p(x_2|q)$.

(b) Fix a joint distribution $p(q)p(x_1|q)p(x_2|q)$. First, the sum rate constraint is easy 
to establish since
\[
I(X_1; Y_1|X_2, Q) + I(X_2; Y_1|Q) = I(X_1, X_2; Y_1|Q),
\]
\[
I(X_1; Y_2|Q) + I(X_2; Y_2|X_1, Q) = I(X_1, X_2; Y_2|Q).
\]
Note that $X_2$ is conditionally independent of $X_1$ given $Q$. Thus, $I(X_1; Y_2|Q) \leq \ 
I(X_1; Y_2|X_2, Q)$ and $I(X_2; Y_1|Q) \leq I(X_2; Y_1|X_1, Q)$. This shows that the rate 
region in (a) is always contained in the simultaneous decoding inner bound.

2. Problem 6.7.

(a) The achievable rate is given as follows
\[
R_1 < \min\left\{ C\left(\frac{I_2}{1+S_2}\right), C(S_1) \right\},
\]
\[
R_2 < \min\left\{ C\left(\frac{I_1}{1+S_1}\right), C(S_2) \right\}.
\]

(b) The simultaneous nonunique decoding inner bound is
\[
R_1 < C(S_1),
\]
\[
R_2 < C(S_2),
\]
\[
R_1 + R_2 < \min\{C(S_1 + I_1), C(S_2 + I_2)\}.
\]
When $I_1/(1+S_1) \geq S_2$ and $I_2/(1+S_2) \geq S_1$, which is equivalent to the very strong 
interference conditions with Gaussian input, both regions reduce to $R_1 < C(S_1)$ 
and $R_2 < C(S_2)$.

(c) Suppose $g_{21} = g_{12} = 0$. Then, using successive cancellation, $R_1 = R_2 = 0$. In 
contrast, using simultaneous nonunique decoding, all rate pairs $(R_1, R_2)$ such that 
$R_1 + R_2 \leq \min\{C(S_1), C(S_2)\}$ can be achieved.
For the first channel, the condition $I > S$ implies the signals operate under the strong interference regime. Hence the capacity region is the set of rate pairs $(R_1, R_2)$ such that

\begin{align*}
R_1 &\leq C(S), \\
R_2 &\leq C(S), \\
R_1 + R_2 &\leq C(S + I).
\end{align*}

For the second channel, from the simultaneous nonunique decoding inner bound, the following rate region is achievable

\begin{align*}
R_1 &\leq C(I), \\
R_2 &\leq C(I), \\
R_1 + R_2 &\leq C(S + I).
\end{align*}

Since $I > S$ and thus $C(I) > C(S)$, the second channel has a larger capacity region.

(a) Suppose that for encoder 1, the first fraction $\alpha$ of the time expends a power $\beta_1 P_1$ and the second fraction $\bar{\alpha}$ of the time uses a power $\tilde{\beta}_1 P_1$.
Therefore, for encoder 1, the SNR for fraction $\alpha$ of the time is $\beta_1 S_1$ and the INR it causes at decoder 2 is $\beta_1 I_2$. The corresponding terms for the second encoder are $\beta_2 S_2$ and $\beta_2 I_1$ respectively.
Let $\tilde{\beta}_1 S_1, \tilde{\beta}_1 I_2, \tilde{\beta}_2 S_1, \tilde{\beta}_2 I_1$, be the corresponding terms in the second time slot $\bar{\alpha}$.
Since the average power constraint $P_1$ must be met, we have

\[
\alpha \beta_1 P_1 + \bar{\alpha} \tilde{\beta}_1 P_1 = P_1
\]

Solving for $\tilde{\beta}_1$ gives us $\tilde{\beta}_1 = (1 - \alpha \beta_1)/\bar{\alpha}$. We obtain a similar expression for $\tilde{\beta}_2$.
Now we can write the capacity region as the set of $(R_1, R_2)$ such that

\begin{align*}
R_1 &< \alpha C \left( \frac{\beta_1 S}{1 + \beta_2 I} \right) + \bar{\alpha} C \left( \frac{(1 - \alpha \beta_1) S}{\bar{\alpha} + (1 - \alpha \beta_2) I} \right), \\
R_2 &< \alpha C \left( \frac{\beta_2 S}{1 + \beta_1 I} \right) + \bar{\alpha} C \left( \frac{(1 - \alpha \beta_2) S}{\bar{\alpha} + (1 - \alpha \beta_1) I} \right)
\end{align*}
(b) The SND region reduces to the set of \((R_1, R_2)\) such that
\[
\begin{align*}
R_1 &< \alpha C (\beta_1 S) + \bar{\alpha} C \left( \frac{1 - \alpha \beta_1}{\bar{\alpha}} S \right), \\
R_2 &< \alpha C (\beta_2 S) + \bar{\alpha} C \left( \frac{1 - \alpha \beta_2}{\bar{\alpha}} S \right), \\
R_1 + R_2 &< \alpha C (\beta_1 S + \beta_2 I) + \bar{\alpha} C \left( \frac{(1 - \alpha \beta_1) S + (1 - \alpha \beta_2) I}{\bar{\alpha}} \right), \\
R_1 + R_2 &< \alpha C (\beta_2 S + \beta_1 I) + \bar{\alpha} C \left( \frac{(1 - \alpha \beta_2) S + (1 - \alpha \beta_1) I}{\bar{\alpha}} \right).
\end{align*}
\]

5. Problem 6.13. For achievability, taking \(U_1 = T\) and \(U_2 = \emptyset\), the Han–Kobayashi inner bound reduces to the set of rate pairs \((R_1, R_2)\) such that
\[
\begin{align*}
R_1 &< I(X_1; Y_1 | Q), \\
R_2 &< H(Y_2 | T, Q), \\
R_1 + R_2 &< I(X_1; Y_1 | T, Q) + H(Y_2 | Q)
\end{align*}
\]
for some pmf \(p(q)p(x_1|q)p(x_2|q)\).

The converse is as follows. Let \(Q \sim \text{Unif}[1:n]\) be independent of \((X_{1}^{n}, X_{2}^{n}, T^{n}, Y_{1}^{n}, Y_{2}^{n})\) and identify \((X_{1Q}, T_{Q}, Y_{1Q}, Y_{2Q}) = (X_{1}, T, Y_{1}, Y_{2})\). Then,
\[
\begin{align*}
nR_1 &\leq I(M_1; Y_1^n) + n\epsilon_n \\
&= I(X_1^n; Y_1^n) + n\epsilon_n \\
&\leq \sum_{i=1}^{n} I(X_{1i}; Y_{1i}) + n\epsilon_n \\
&= \sum_{i=1}^{n} I(X_{1i}; Y_{1i}| Q = i) + n\epsilon_n \\
&= nI(X_1; Y_1 | Q) + n\epsilon_n.
\end{align*}
\]
For the second inequality, consider

\[ nR_2 \leq I(M_2; Y^n_2) + n\epsilon_n \]

\[ \leq I(M_2; Y^n_2 | M_1) + n\epsilon_n \]

\[ = I(M_2, X^n_2; Y^n_2 | M_1, T^n) + n\epsilon_n \]

\[ \leq I(M_1, M_2, X^n_2; Y^n_2 | T^n) + n\epsilon_n \]

\[ = H(Y^n_2 | T^n) + n\epsilon_n \]

\[ \leq \sum_{i=1}^{n} H(Y_{2i} | T_i) + n\epsilon_n \]

\[ = \sum_{i=1}^{n} H(Y_{2i} | T_i, Q = i) + n\epsilon_n \]

\[ = nH(Y_2 | T, Q) + n\epsilon_n, \]

where (a) follows since \( M_1 \) and \( M_2 \) are independent and (b) follows since \( Y^n_2 \) is a function of \( (T^n, X^n_2) \). For the last inequality, consider

\[ n(R_1 + R_2) \leq I(X^n_1; Y^n_1) + I(X^n_2; Y^n_2) + n\epsilon_n \]

\[ \leq I(X^n_1; T^n, Y^n_1) + I(X^n_2; Y^n_2) + n\epsilon_n \]

\[ = I(X^n_1; T^n) + I(X^n_1; Y^n_1 | T^n) + I(X^n_2; Y^n_2) + n\epsilon_n \]

\[ \leq H(T^n) + H(Y^n_1 | T^n) - H(Y^n_1 | X^n_1, T^n) + H(Y^n_2) - H(Y^n_2 | X^n_2) + n\epsilon_n \]

\[ \leq \sum_{i=1}^{n} (H(Y_{1i} | T_i) - H(Y_{1i} | X_{1i}) + H(Y_{2i})) + n\epsilon_n \]

\[ = n(H(Y_1 | T, Q) - H(Y_1 | X_1, Q) + H(Y_2 | Q)) + n\epsilon_n \]

\[ = n(I(X_1; Y_1 | T, Q) + H(Y_2 | Q)) + n\epsilon_n, \]

where (c) follows since \( H(Y^n_2 | X^n_2) = H(T^n) \) and \( T^n \rightarrow X^n_1 \rightarrow Y^n_1 \) form a Markov chain. Taking \( n \rightarrow \infty \) completes the proof of the converse.