Communication of a DMS over a DMC

- DMS $U$ with entropy $H(U)$
- DMC $p(y|x)$ with capacity $C$
- A $(|U|^k, n)$ joint source–channel code of rate $r = k/n$ symbols/transmission:
  - Encoder: $x^n(u^k) \in X^n$
  - Decoder: $\hat{u}^k(y^n) \in \hat{U}^k$
- $r$ achievable if $\exists (|U|^r, n)$ codes such that $\lim_{n \to \infty} P\{\hat{U}^r \neq U^r\} = 0$
- What is the necessary and sufficient condition for $r$ to be achievable?
Joint source–channel coding

Source–channel separation theorem (Shannon 1959)

- If \( rH(U) < C \), then \( r \) is achievable
- If \( r \) is achievable, then \( rH(U) \leq C \)

\[
H(U) \leq \frac{1}{k} I(U_k; \hat{U}_k) \\
\leq \frac{1}{k} I(U_k; Y^n) \\
\leq \frac{1}{k} \sum_{i=1}^{n} I(X_i; Y_i) \leq \frac{1}{r} C
\]

Proof of achievability

- Separate source and channel coding is asymptotically optimal
  - \( P\{\hat{U}_r \neq U_r \} \leq P\{\hat{U}_r \neq U_r, \hat{M} = M\} + P\{\hat{M} \neq M\} \)
- Source coding: \( P\{\hat{U}_r \neq U_r, \hat{M} = M\} \to 0 \) if \( R > H(U) \) bits/symbol
- Channel coding: \( P\{\hat{M} \neq M\} \to 0 \) if \( rR < C \)
- Combined together, \( P\{\hat{U}_r \neq U_r\} \to 0 \) if \( rH(U) < C \)
- Basis for digital communication: Bits as “universal” source–channel interface
Communication of a 2-DMS over a DM-MAC

- 2-DMS \((U_1, U_2)\) and a DM-MAC \(p(y|x_1, x_2)\)
- A \((|U_1|^{k_1}, |U_2|^{k_2}, n)\) joint source–channel code of rate pair \((r_1 = k_1/n, r_2 = k_2/n)\):
  - Encoder \(j = 1, 2: x_j^n(u_j^{k_j})\)
  - Decoder: \(\hat{U}_1^{k_1}(y^n), \hat{U}_2^{k_2}(y^n)\)
- Probability of error: \(P_e(n) = P\{ (\hat{U}_1^{k_1}, \hat{U}_2^{k_2}) \neq (U_1^{k_1}, U_2^{k_2}) \}\)
- Lossless communication if \(\exists (|U_1|^{k_1}, |U_2|^{k_2}, n)\) codes with \(\lim_{n \to \infty} P_e(n) = 0\)
- What is the necessary and sufficient condition for lossless communication?
- For simplicity, assume that \(r_1 = r_2 = 1\) symbol/transmission

A separate source and channel coding scheme

- \(\mathcal{C}\) of the DM-MAC is the set of \((R_1, R_2)\) such that
  \[
  R_1 \leq I(X_1; Y|X_2, Q), \\
  R_2 \leq I(X_2; Y|X_1, Q), \\
  R_1 + R_2 \leq I(X_1, X_2; Y|Q)
  \]
  for some \(p(q)p(x_1|q)p(x_2|q)\)
- \(\mathcal{R}^*\) for distributed lossless compression is the set of \((R_1, R_2)\) such that
  \[
  R_1 \geq H(U_1|U_2), \\
  R_2 \geq H(U_2|U_1), \\
  R_1 + R_2 \geq H(U_1, U_2)
  \]
- Hence, separation achieves lossless communication if \(\mathcal{C} \cap \mathcal{R}^*\) is not empty:
  \[
  H(U_1|U_2) < I(X_1; Y|X_2, Q), \\
  H(U_2|U_1) < I(X_2; Y|X_1, Q), \\
  H(U_1, U_2) < I(X_1, X_2; Y|Q)
  \]
  for some \(p(q)p(x_1|q)p(x_2|q)\)
**Examples**

- **MAC with orthogonal components** \( p(y|x_1, x_2) = p(y_1|x_1)p(y_2|x_2): \)
  
  Lossless communication is possible iff
  
  \[
  \begin{align*}
  H(U_1|U_2) &\leq C_1, \\
  H(U_2|U_1) &\leq C_2, \\
  H(U_1, U_2) &\leq C_1 + C_2
  \end{align*}
  \]

- **Independent sources** \( (U_1, U_2) \sim p(u_1)p(u_2): \)
  
  Lossless communication is possible iff
  
  \[
  \begin{align*}
  H(U_1) &\leq I(X_1; Y|X_2, Q), \\
  H(U_2) &\leq I(X_2; Y|X_1, Q), \\
  H(U_1) + H(U_2) &\leq I(X_1, X_2; Y|Q),
  \end{align*}
  \]
  
  for some \( p(q)p(x_1|q)p(x_2|q) \)

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**Counterexample (Cover–El Gamal–Salehi 1980)**

- Let \( (U_1, U_2) \) be a 2-DMS with \( p_{U_1,U_2}(0, 0) = p_{U_1,U_2}(0, 1) = p_{U_1,U_2}(1, 1) = 1/3 \)
- Send \( (U_1, U_2) \) over a binary erasure MAC \( (X_1, X_2 \in \{0, 1\}, Y = X_1 + X_2) \)
- \( \mathcal{R}^* \cap \mathcal{C} = \emptyset \)

- Cannot send \( (U_1, U_2) \) losslessly using this separate source and channel scheme

- Now consider uncoded transmission: \( X_{1i} = U_{1i}, X_{2i} = U_{2i}, i \in [1 : n] \)
- Hence, joint source–channel coding achieves error-free transmission!
- Separation does not hold in general for sending sources over multiuser channels
A joint source–channel coding scheme

The 2-DMS \((U_1, U_2)\) can be sent losslessly over a DM-MAC \(p(y|x_1, x_2)\) if

\[
\begin{align*}
H(U_1 | U_2) &< I(X_1; Y | X_2, U_2, Q), \\
H(U_2 | U_1) &< I(X_2; Y | X_1, U_1, Q), \\
H(U_1, U_2) &< I(X_1, X_2; Y | Q)
\end{align*}
\]

for some \(p(q, x_1, x_2 | u_1, u_2) = p(q)p(x_1 | u_1, q)p(x_2 | u_2, q)\)

Special cases:

- **Separate source and channel coding**: Set \(p(x_1 | u_1, q)p(x_2 | u_2, q) = p(x_1 | q)p(x_2 | q)\)
- **Counterexample**: Set \(Q = \emptyset, X_1 = U_1,\) and \(X_2 = U_2\)

Proof of achievability

- Assume \(|Q| = 1\) (the general case follows by coded time sharing)
- **Codebook generation**: Fix \(p(x_1 | u_1)\) and \(p(x_2 | u_2)\)
  - For each \(u_1^n \in U_1^n\), randomly and independently generate \(x_1^n(u_1^n) \sim \prod_{i=1}^n P_{X_1|U_1}(x_1 | u_1\_i)\)
  - For each \(u_2^n \in U_2^n\), randomly and independently generate \(x_2^n(u_2^n) \sim \prod_{i=1}^n P_{X_2|U_2}(x_2 | u_2\_i)\)
- **Encoding**:
  - Upon observing \(u_1^n\), encoder 1 transmits \(x_1^n(u_1^n)\)
  - Upon observing \(u_2^n\), encoder 2 transmits \(x_2^n(u_2^n)\)
  - No more than \(2^n(H(U_1, U_2) + \delta(\varepsilon))\) codeword pairs \((x_1^n, x_2^n)\) can simultaneously occur w.h.p.
- **Decoding**:
  - Find the unique pair \((\hat{u}_1^n, \hat{u}_2^n)\) such that \((\hat{u}_1^n, \hat{u}_2^n, x_1^n(\hat{u}_1^n), x_2^n(\hat{u}_2^n), y^n) \in T^{(n)}_{\varepsilon}\)
Analysis of the probability of error

- Consider the error events:
  \[ E_1 = \{(U_1^n, U_2^n, X_1^n(U_1^n), X_2^n(U_2^n), Y^n) \notin \mathcal{T}_{\epsilon}^{(n)} \} \]
  \[ E_2 = \{(\tilde{u}_1^n, U_2^n, X_1^n(\tilde{u}_1^n), X_2^n(U_2^n), Y^n) \in \mathcal{T}_{\epsilon}^{(n)} \text{ for some } \tilde{u}_1^n \neq U_1^n \} \]
  \[ E_3 = \{(U_1^n, \tilde{u}_2^n, X_1^n(U_1^n), X_2^n(\tilde{u}_2^n), Y^n) \in \mathcal{T}_{\epsilon}^{(n)} \text{ for some } \tilde{u}_2^n \neq U_2^n \} \]
  \[ E_4 = \{(\tilde{u}_1^n, \tilde{u}_2^n, X_1^n(\tilde{u}_1^n), X_2^n(\tilde{u}_2^n), Y^n) \in \mathcal{T}_{\epsilon}^{(n)} \text{ for some } \tilde{u}_1^n \neq U_1^n, \tilde{u}_2^n \neq U_2^n \} \]

- Then, the average probability of error
  \[ P(E) \leq P(E_1) + P(E_2) + P(E_3) + P(E_4) \]

- Cannot use the packing lemma or joint typicality lemma to bound \( P(E_j) \), \( j = 2, 3, 4 \)

- Use basic properties of joint typicality (see NIT 14.1.2)

Suboptimality of the scheme

- Let \( U_1 = U_2 = U \)

- The scheme yields:
  \[ H(U) < \max_{p(x_1|u)p(x_2|u)} I(X_1, X_2; Y) \]
  Cannot generate all joint pmfs on \((X_1, X_2)\) in general

- But, since both senders observe same source, can use cooperative coding:
  \[ H(U) < \max_{p(x_1, x_2)} I(X_1, X_2; Y) \]
  which can be less stringent than using the joint source-channel coding scheme

- In general the scheme can be improved when \((U_1, U_2)\) have a common part
Common part of a 2-DMS

- Let \((U_1, U_2) \sim p(u_1, u_2)\). Arrange \(p(u_1, u_2)\) in largest block diagonal form:

\[
\begin{array}{cccc}
  u_0 = 1 & 0 & \cdots & 0 \\
  0 & u_0 = 2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & u_0 = k \\
\end{array}
\]

- Common part between \(U_1\) and \(U_2\): \(U_0 = u_0\) if \((U_1, U_2)\) is in block \(u_0 \in [1 : k]\)

- Formally (Gács–Körner 1973, Witsenhausen 1975):
  Let \(g_j : U_j \rightarrow [1 : k], j = 1, 2\), be functions with largest \(k\) such that:

\[
P\{g_j(U_j) = u_0\} > 0 \text{ for every } u_0 \in [1 : k], \ j = 1, 2, \text{ and } P\{g_1(U_1) = g_2(U_2)\} = 1
\]

Then the common part is \(U_0 = g_1(U_1) = g_2(U_2)\)

Common part of a 2-DMS \((U_1, U_2)\)

- Example:

\[
\begin{array}{cccc|ccc}
  u_1 & u_0 = 1 & u_0 = 2 \\
  u_2 & 1 & 2 & 3 & 4 \\
  & 1 & 0.1 & 0.2 & 0 & 0 \\
  & 2 & 0.1 & 0.1 & 0 & 0 \\
  & 3 & 0.1 & 0.1 & 0 & 0 \\
  & 4 & 0 & 0 & 0.2 & 0.1 \\
\end{array}
\]

Here \(k = 2\), \(P\{U_0 = 1\} = 0.7\), \(P\{U_0 = 2\} = 0.3\)

- Now, consider a 2-DMS \((U_1, U_2)\)

- Common part between \(U_1^n\) and \(U_2^n\) is \(U_0^n\) (up to relabeling)

- Hence, the DMS \(U_0\) is the common part of the 2-DMS \((U_1, U_2)\)

- Can modify our joint source–channel scheme to include common part
Three-index separate source and channel coding scheme

- Source coding:

```
U^n_1 \rightarrow \text{Encoder 1} \rightarrow M_1
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U^n_2 \rightarrow \text{Encoder 2} \rightarrow M_0, M_2
```

- \( M_0 \) can depend only on \( U^n_0 \)

- Hence, the optimal rate region \( \mathcal{R}^* \) is the set of \((R_0, R_1, R_2)\) such that

\[
R_1 \geq H(U_1 | U_2),
R_2 \geq H(U_2 | U_1),
R_1 + R_2 \geq H(U_1, U_2 | U_0),
R_0 + R_1 + R_2 \geq H(U_1, U_2)
\]

- Lossless compression of \( U_0 \); S–W coding of \((U_1, U_2)\) for every \( u^n_0 \)

Three-index separate source and channel coding scheme

- Channel coding:

```
M_0, M_1 \rightarrow \text{Encoder 1} \rightarrow X^n_1 \rightarrow p(y|x_1, x_2) \rightarrow Y^n \rightarrow \text{Decoder} \rightarrow \hat{M}_0, \hat{M}_1, \hat{M}_2
```

- The capacity region \( \mathcal{C} \) with a common message is the set of \((R_0, R_1, R_2)\) such that

\[
R_1 \leq I(X_1; Y | X_2, W),
R_2 \leq I(X_2; Y | X_1, W),
R_1 + R_2 \leq I(X_1, X_2; Y | W),
R_0 + R_1 + R_2 \leq I(X_1, X_2; Y)
\]

for some \( p(w)p(x_1 | w)p(x_2 | w) \) with \(|\mathcal{W}| \leq \min\{|\mathcal{X}_1| \cdot |\mathcal{X}_2| + 2, |\mathcal{Y}| + 3\} \) (Problem 5.19)

- Use superposition coding
Three-index separate source and channel coding scheme

- Hence, separation achieves lossless communication if $C \cap R^*$ is not empty:

$$H(U_1 | U_2) < I(X_1; Y | X_2, W),$$
$$H(U_2 | U_1) < I(X_2; Y | X_1, W),$$
$$H(U_1, U_2 | U_0) < I(X_1, X_2; Y | W),$$
$$H(U_1, U_2) < I(X_1, X_2; Y)$$

for some $p(w)p(x_1 | w)p(x_2 | w)$ with $|W| \leq \min\{|X_1| \cdot |X_2| + 2, |Y| + 3\}$

A joint source–channel coding scheme with common part

- Combine three-index coding scheme and Theorem 14.1:


A 2-DMS with common part can be sent losslessly over a DM-MAC if

$$H(U_1 | U_2) < I(X_1; Y | X_2, U_2, W),$$
$$H(U_2 | U_1) < I(X_2; Y | X_1, U_1, W),$$
$$H(U_1, U_2 | U_0) < I(X_1, X_2; Y | U_0, W),$$
$$H(U_1, U_2) < I(X_1, X_2; Y)$$

for some $p(w)p(x_1 | u_1, w)p(x_2 | u_2, w)$

- $U_0$ is represented by $W$, chosen to maximize cooperation between the senders
- Use the previous joint source–channel scheme for each $w$
- Optimal for the DM-MAC with common message (Slepian–Wolf 1973)
- Not optimal in general (Dueck 1981)
Summary

- Source channel separation holds for point to point communication
- Source–channel separation does not hold in general for multiuser channels
- Joint source–channel coding schemes that utilize the correlation between the sources for cooperative transmission
- Common part of a 2-DMS

References


