EE378B Inference, Estimation, and Information Processing

Spectral graph localization

Andrea Montanari *Lecture 11-12 - Due on 3/1/2021*

Homework should be submitted via Gradescope, by Monday afternoon (unless Monday is a holiday): the code will be communicated by an announcement on Canvas.

For getting credit for the class, you are required to present solutions of some of these homeworks during the first 15 minutes of class starting on 1/20. Please, sign up for (at least) one slot, and be sure that your explanation lasts 15 minutes (or less). For these presentations, you are free to choose whatever format you prefer (slides, typed notes, handwriting, ...).

This week, the presentations will be:

- Monday: Question (1).
- Wednesday: Questions (2), (3), (4), (5).

Problem

The MDS-MAP algorithm estimates the positions of n objects $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ from pairwise distance measurements. Namely, for a graph $G = (V = [n], E)$ we are given a noise measurement $\tilde{d}_{ij}$ of $\|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2$ for any $(i, j) \in E$. The algorithm is outlined below and reconstructs the positions up to a rigid motion (rotation and translation).

MDS-MAP

Input : Graph $G = (V, E)$. Distance measurements $\tilde{d}_{ij}$, $(i, j) \in E$
Output : Low-dimensional coordinates $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$

1: Estimate $\tilde{d}_{ij}$, $(i, j) \notin E$ by computing the shortest path between i and j in G with edge lengths $\tilde{d}_{ij}$;
2: Construct the matrix $\tilde{\boldsymbol{D}} \in \mathbb{R}^{n\times n}$ by letting $\tilde{D}_{ij} = \tilde{d}_{ij}^2$;
3: Centering $\boldsymbol{Q} = -\boldsymbol{P}^\perp \tilde{\boldsymbol{D}} \boldsymbol{P}^\perp$, $\boldsymbol{P}^\perp = \boldsymbol{I} - \boldsymbol{1}\boldsymbol{1}^\mathsf{T}/n$;
4: Compute the eigen-decomposition $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^\mathsf{T}$;
5: Project onto the d eigen-vectors $\boldsymbol{X} = \boldsymbol{U}_d \boldsymbol{\Sigma}_d^{1/2}$;
6: Return the rows of $\boldsymbol{X} = [\boldsymbol{x}_1, \dots, \boldsymbol{x}_n]^\mathsf{T}$.

For clarity, in step 1 we estimate

$$\tilde{d}_{ij} := \min\Big\{ \sum_{(u,v)\in\gamma} \tilde{d}_{uv} : \;\; \gamma \text{ a path in } G \text{ from } u \text{ to } v \Big\} \tag{1}$$

The file cities.txt contains the geographic coordinates (latitude north and longitude west) of 130 cities in US and Canada, and will be used for this homework.

(1) Explain heuristically the rationale for the algorithm described above.

(2) Using the fact that the Earth's equatorial radius is $6,378,137$ and approximating the Earth as a perfect sphere, compute the distances of each pair of cities. (By this we mean the distance to be walked going straight on the Earth surface between one city and the other).

(3) Construct a graph $G = (V, E)$ with vertices corresponding to the cities, and an edge going from each city to its 6 closest neighbors.
[For points (1) and (2) above, you are requested to submit only the code for doing the computations.]

(4) We want to reconstruct the cities positions from the graph G only, where you set $\tilde{d}_{ij} = 1$ for $(i, j) \in E$. Apply the MDS-MAP algorithm to this graph and estimate the positions in two dimensions for the 130 cities. Present a plot of these positions.

(5) Repeat the calculation at point (3) by using MDS-MAP with the fully connected graph with edge lengths as computed in point (1). Also in this case use $d = 2$ in the algorithm.