Homework 4

Problem 1

For $M \in \mathbb{R}^{m \times n}$, we want to prove properties of the operator norm (w.r.t. Euclidean norm).

Part 1

Prove that operator norm admits following characterizations if $M$ has singular values:

$$\sigma_1(M) \geq \sigma_2(M) \geq \ldots \geq \sigma_n(M)$$

$$\|M\|_{OP} = \max \left\{ \langle x, My \rangle : \|x\|_2 = \|y\|_2 = 1 \right\} = \sigma_1(M)$$

Proof. Let us start with the first part. This just follows from the self-duality relationship of the Euclidean Norm, i.e. $\|\cdot\|_2,^* = \|\cdot\|_2$. This duality just means that for any $u \in \mathbb{R}^m$:

$$\|u\|_2 = \sup_{x: \|x\|_2 = 1} \{ \langle x, u \rangle \} = \max_{x: \|x\|_2 = 1} \{ \langle x, u \rangle \}$$

(Sometimes the above is phrased over $x$ in the unit ball, i.e. the supremum is taken over $x, \|x\|_2 \leq 1$, but one may easily check that these are the same, since if the norm of $x \not\geq 1$, one can make the quantity larger in absolute value by rescaling $x$ to norm 1).

We apply this with $u = My$ below, as follows:

$$\max \left\{ \langle x, My \rangle : \|x\|_2 = \|y\|_2 = 1 \right\} = \max_{\|y\|_2 = 1} \left\{ \max_{\|x\|_2 = 1} \langle x, My \rangle \right\}$$

$$= \max_{\|y\|_2 = 1} \|My\|_2$$

$$= \|M\|_{OP}$$

The last step is just the definition of operator norm.

The second result (i.e. that the above is $\sigma_1(M)$) just follows from the variational property of the first singular value and the 1st result.
Part 2
To prove that $\|M\|_{OP} = \|M^T\|_{OP}$ and $\|AB\|_{OP} \leq \|A\|_{OP} \|B\|_{OP}$.

Proof. For Part 1 (second characterization above), it suffices to recall $\sigma_1(M) = \sigma_1(M^T)$.

Alternatively, we may use the first characterization from Part 1, and recall by adjointness:

$$\langle x, My \rangle = \langle y, M^T x \rangle$$

and thus:

$$\max \{ \langle x, My \rangle, \|x\|_2 = \|y\|_2 = 1 \} = \max \{ \langle y, M^T x \rangle, \|x\|_2 = \|y\|_2 = 1 \}$$

For the second statement: Take $v$ arbitrary with $\|v\|_2 = 1$ and $Bv \neq 0$. Also define $u = \frac{Bv}{\|Bv\|_2}$ and note that $\|u\|_2 = 1$.

Then:

$$\|ABv\|_2 = \|Bv\|_2 \|Au\|_2 \leq \|B\|_{op} \|A\|_{op}$$

Here we used that $\|v\|_2 = \|u\|_2 = 1$ and the definition of the operator norm. If $Bv = 0$ then $\|ABv\|_2 = 0$ and the above is still true. Hence taking supremum over all $v, \|v\|_2 = 1$, we see:

$$\|AB\|_{op} = \sup_{v: \|v\|_2 = 1} \|ABv\|_2 \leq \|B\|_{op} \|A\|_{op} = \|A\|_{op} \|B\|_{op}$$

Part 3
Prove that $\|\cdot\|_{OP}$ is indeed a norm.

Proof. (i) Fix any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then:

$$\|\alpha M\|_{op} = \sup_{x: \|x\|_2 = 1} \{ \|\alpha Mx\|_2 \} = \sup_{x: \|x\|_2 = 1} \{ |\alpha| \|Mx\|_2 \} = |\alpha| \sup_{x: \|x\|_2 = 1} \{ \|Mx\|_2 \} = \alpha \|M\|_{op}$$

First + last equality are definition. Second inequality is that $(\mathbb{R}^n, \|\cdot\|_2)$ is normed space, third inequality is property of suprema.

(ii) Let us now prove the triangle inequality. Fix any $x \in \mathbb{R}^n, \|x\|_2 = 2$, then:

$$\|(A + B)x\|_2 = \|Ax + Bx\|_2 = \|Ax\|_2 + \|Bx\|_2 \leq \|A\|_{op} + \|B\|_{op}$$

The first inequality follows by the triangle inequality in $(\mathbb{R}^n, \|\cdot\|_2)$ and the second by definition of operator norm, since $\|x\|_2 = 1$. Now take the supremum over all $x, \|x\|_2 = 1$, to conclude that:

$$\|AB\|_{op} = \sup_{x: \|x\|_2 = 1} \|(A + B)x\|_2 \leq \|A\|_{op} + \|B\|_{op}$$

(iii) Now, if $M = 0$, then for any $x$ it holds that $Mx = 0$ and thus $\|M\|_{OP} = 0$. For the other direction, let $\|M\|_{OP} = 0$. For the sake of contradiction, assume that there exists a column $j$, such that $M_{j} \neq 0$. Then (with $e_j$ the j-th unit vector):
The last step follows because \( \|\cdot\|_2 \) is a norm in \( \mathbb{R}^n \). But since \( \|e_j\|_2 = 1 \) this means that \( \|M\|_{OP} \geq \|M \cdot e_j\|_2 > 0 \). Which is a contradiction. Thus \( M = 0 \).

\[ \square \]

**Problem 2**

We have a matrix \( X \in \mathbb{R}^{d_1 \times d_2} \) with i.i.d. \( b \)-subexponential entries and want to bound its operator norm. For integer \( L \), define:

\[
S_L = \left\{ 0, 1, \frac{1}{2}, \ldots, \frac{1}{2L} \right\}
\]

Let:

\[
N^n(L) = \left\{ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1, x^2_i \in S_L \right\}
\]

We also define the function: \( \pi_{<\ell}(x)_i = x_i 1 (x_i^2 > 2^{-\ell}) \). And then \( N^n_{<\ell}(L) = \pi_{<\ell}(N^n(L)) \).

Similarly with \( \pi_{=\ell} \).

**Part a)**

Give example of sub-exponential RV that is not sub-Gaussian.

**Proof.** Well take \( X \sim Exp(1) \), so that it has pdf \( f(x) = \exp(-x)\mathbf{1}_{(0, \infty)}(x) \) and CDF \( F(x) = 1 - \exp(-x) \). The claim is then that the center \( Y = X - 1 \), where \( \mathbb{E}[X] = 1 \) satisfies the required properties. We get for \( \lambda < 1 \) that:

\[
\mathbb{E}[\exp(\lambda(X - 1))] = \frac{1}{1 - \lambda} \exp(-\lambda) = \exp(- \log(1 - \lambda) - \lambda)
\]

Notice that this is \( \infty \) for \( \lambda > 1 \), hence \( Y \) cannot be subgaussian.

Next, recall the inequality (e.g. John Duchi’s STATS311 lecture notes):

\[
- \log(1 - \lambda) \leq \lambda + 2\lambda^2, \quad 0 \leq \lambda \leq \frac{1}{2}
\]

This gives us for \( \lambda \leq \frac{1}{2} \):

\[
\mathbb{E}[\exp(\lambda(X - 1))] \leq \exp(2\lambda^2)
\]

Thus \( Y = X - 1 \) is subexponential.
Part b)

Prove that for \( L = \log_2(n) + c_0 \), \( N(L) \) is an \( \epsilon_0 \)-net of \( B_2^n(0, 1) \) for some \( \epsilon_0 < 1/2 \). Thus, for suitably chosen \( L \) it holds that:

\[
P \left[ \|X\|_{\infty} \geq t \right] \leq P \left[ \max_{u \in N^{d_1}(L)} \max_{v \in N^{d_2}(L)} |\langle u, Xv \rangle| \geq C(\epsilon_0)t \right]
\]

Proof. For the first part, take any \( x \in B_2^n(0, 1) \). Without loss of generality we may assume that \( x_i \geq 0 \) for all \( i \) (the argument below will otherwise just need some sign flips). Then for the \( i \)-th coordinate, let:

\[
\alpha_i^2 = \sup \{ u \in S_L : u \leq x_i^2 \}
\]

Then \( \alpha = (\alpha_1, \ldots, \alpha_n) \in N(L) \), by construction (the norm constraint is also satisfied since \( \alpha_i^2 \leq x_i^2 \) elementwise and so \( \|\alpha\|_2 \leq \|x\|_2 \leq 1 \)). How far can \( \alpha \) be from \( x \) in the Euclidean norm? We need to argue (for some \( c_0 \) which we will pick), that it can be at most \( \epsilon_0 < 1/2 \).

To this end, let us fix a coordinate \( i \) and first consider the case \( x_i^2 \in [2^{-(L-1)}, 2^{-L}] \) for \( L \) in our grid. Then \( \alpha_i = 2^{-(L-1)/2} \), while \( x_i \in [2^{-(L-1)/2}, 2^{-L/2}] \). This shows that:

\[
|\alpha_i - x_i| \leq \left( \sqrt{2} - 1 \right) 2^{-(L-1)/2} \leq \left( \sqrt{2} - 1 \right) |x_i|
\]

And thus with \( c = \left( \sqrt{2} - 1 \right)^2 \)

\[
(\alpha_i - x_i)^2 \leq cx_i^2
\]

What remains to control is what happens for small \( x_i \), i.e. with \( x_i^2 < 2^{-L} \). Then we may use the bound: \( (\alpha_i - x_i)^2 \leq 2^{-L} \) and further instead of \( c_0 \) let us specify \( c_1 = 2^{c_0} \). Then the above bound with \( L = \log_2(n) + c_0 \) let results in \( \leq \frac{n}{\alpha^2} \). Combining everything we get:

\[
\sum_i (\alpha_i - x_i)^2 = \sum_{i: x_i^2 > 2^{-L}} (\alpha_i - x_i)^2 + \sum_{i: x_i^2 \leq 2^{-L}} (\alpha_i - x_i)^2
\]

\[
\leq \sum_{i: x_i^2 > 2^{-L}} cx_i^2 + \sum_{i: x_i^2 \leq 2^{-L}} (\alpha_i - x_i)^2 \frac{1}{nc_1}
\]

\[
\leq c \sum_i x_i^2 + \sum_i \frac{1}{nc_1}
\]

\[
\leq c + \frac{1}{c_1}
\]

To conclude we pick \( c_0 > 0 \), so that \( 1/c_1 \) is small enough, so that the whole expression becomes \( < 1/4 \) (recall we specified \( c \) explicitly).

For the second part, we may just use Lemma 3.1.5. from the lecture notes, which even establishes the form:

\[
C(\epsilon_0) = \frac{1}{1 - 2\epsilon_0}
\]
Part c)

Prove that for $c_1$ a suitable constant,

$$\max \{|N^n_{=\ell}|, |N^n_{<\ell}|\} \leq \left(\frac{c_1 n}{2^\ell}\right)^2$$

Proof. Let $k = 2^\ell$. Let us study $|N^n_{=\ell}|$ first. Each coordinate can be either $\pm 2^{\ell-1}$ or 0. To satisfy the $\|\cdot\|_2$ constraint, we see that we may pick at most $k$ coordinates to be non-zero (which then may take any of the above 3-values). Hence the size of the set is bounded as:

$$|N^n_{=\ell}| \leq \binom{n}{k} 3^k \leq \left(\frac{3en}{k}\right)^k$$

Recalling definition of $k$ we get the required result with $c_1 = 3e$.

For $|N^n_{<\ell}|$ the argument is analogous but needs a bit more care (there can be at most one coordinate with $x_i^2 = 1$, at most 2 with $x_i^2 = 1/2$,... and at most $2^{\ell-1}$ with $x_i^2 = 2^{-\ell+1}$).

Part d)

Prove for any $t \geq 0$, $u \in B_{d_1}^d(0, 1)$, $v \in B_{d_2}^d(0, 1)$:

$$\mathbb{P} \left[ |\langle u, Xv \rangle| \geq t \right] \leq 2 \exp \left\{ -c_0 \min \left( \frac{t}{b\|u\|_\infty \|v\|_\infty}, \frac{t^2}{b^2} \right) \right\}$$

Proof. First let us note: If $X$ is $b$-sub-exponential, then for any $c$ and $\lambda$ such that $|\lambda c| \leq \frac{1}{b}$, we get:

$$\mathbb{E} \left[ \exp (\lambda cX) \right] \leq \exp \left( \frac{b^2 c^2 \lambda^2}{2} \right)$$

But this just means that $cX$ is $|c| \cdot b$ sub-exponential. Hence for fixed $u, v$, $\langle u, Xv \rangle$ is a sum of terms of the form $u_i X_{ij} v_j$, but these are $|u_i v_j| b$ sub-exponential. Hence if we think of reshaping $X$ to be a vector, we may apply Bernstein’s inequality, after controlling the 2-Norm and Infinity-Norm of the vector with entries $u_i v_j b$.

Two-Norm: $\sum_{i,j} u_i^2 v_j^2 b^2 = b^2 \|uv^T\|_F^2 \leq b^2 \|u\|_2^2 \|v\|_2^2 \leq b^2$

Infinity-Norm: $\max_{i,j} |u_i v_j b| \leq b \|u\|_\infty \|v\|_\infty$.

The conclusion follows from Bernstein.

Part e)

Prove for matrix $M$ and $u \in N^{d_1}(L), v \in N^{d_2}(L)$ it holds that:

$$\langle u, Mv \rangle = \sum_{\ell=0}^L [\langle \pi_{=\ell}(u), M \pi_{=\ell}(v) \rangle + \langle \pi_{=\ell}(u), M \pi_{<\ell}(v) \rangle + \langle \pi_{<\ell}(u), M \pi_{=\ell}(v) \rangle]$$

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Proof. I am assuming throughout that we let $n = d_1 \lor d_2$ and choose $L$ based on that. Let us note that for $u \in N^{d_1}(L)$:

$$u = \sum_{\ell=0}^{L} \pi_{=\ell}(u)$$

This follows directly from the definitions (with slight care to check what happens if $u_i = 0$). The same holds for $v \in N^{d_2}(L)$. Thus expanding both $u, v$ we get:

$$\langle u, Mv \rangle = \sum_{\ell=0}^{L} \sum_{k=0}^{L} \langle \pi_{=\ell}(u), M\pi_{=k}(v) \rangle$$

Next write:

$$\sum_{0 \leq \ell \neq k \leq L} = \sum_{\ell=0}^{L-1} \sum_{k=\ell+1}^{L} + \sum_{k=0}^{L-1} \sum_{\ell=k+1}^{L}$$

To conclude we just need to use:

$$\sum_{k=\ell+1}^{L} \pi_{=k}(u) = \pi_{<\ell}(u)$$

Part f)

Now we want to bound:

$$\mathbb{P} \left[ \max_{u \in N^{d_1}(L), v \in N^{d_2}(L)} \langle \pi_{=\ell}(u), M\pi_{=\ell}(v) \rangle \geq t_\ell \right]$$

Proof. Let us fix $\ell$ and then let us note that by the previous definitions, it holds that:

$$\max_{u \in N^{d_1}(L), v \in N^{d_2}(L)} \langle \pi_{=\ell}(u), M\pi_{=\ell}(v) \rangle = \max_{u \in N^{d_1}_{\ell}(L), v \in N^{d_2}_{\ell}(L)} \langle u, Mv \rangle$$

Hence we next fix $u \in N^{d_1}_{\ell}(L), v \in N^{d_2}_{\ell}(L)$. Notice that:

$$\|u\|_\infty^2, \|v\|_\infty^2 \leq 2^{-\ell}$$

Hence plugging in to d) (ignoring the factor 2 since this goes in one direction), we get:

$$\mathbb{P} \left[ (\pi_{=\ell}(u), M\pi_{=\ell}(v)) \geq t_\ell \right] \leq \exp \left\{ -c_0 \min \left( \frac{2^{\ell} t_\ell^2}{b^2}, \frac{t_\ell^2}{b^2} \right) \right\}$$

Hence by the Bonferroni bound, and the bound from c):
\[
\mathbb{P} \left[ \max_{u \in \mathbb{N}^{d_1}(L), v \in \mathbb{N}^{d_2}(L)} \langle \pi = \ell(u), M\pi = \ell(v) \rangle \geq t \ell \right] \leq \left( \frac{c_1 n}{2^\ell} \right)^{2 \cdot 2^\ell} \exp \left\{ -c_0 \min \left( 2^\ell \frac{t \ell}{b}, \frac{t^2 \ell}{b^2} \right) \right\} = \exp \left\{ 2 \times 2^\ell (\log(c_1 n) - \ell \log(2)) - c_0 \min \left( 2^\ell \frac{t \ell}{b}, \frac{t^2 \ell}{b^2} \right) \right\}
\]

By the same argument we may also bound the other quantities. Let us do the second one. The only difference is that now unfortunately we may no longer upper bound both \( \|u\|_{\infty}, \|v\|_{\infty} \). Instead, we only get:

\[
\|u\|_{\infty}^2 \leq 1, \|v\|_{\infty}^2 \leq 2^{-\ell}
\]

This causes a smaller factor in the exponent, as per the expression below:

\[
\mathbb{P} \left[ \max_{u \in \mathbb{N}^{d_1}(L), v \in \mathbb{N}^{d_2}(L)} \langle \pi > \ell(u), M\pi = \ell(v) \rangle \geq t \ell \right] \leq \left( \frac{c_1 n}{2^\ell} \right)^{2 \cdot 2^\ell} \exp \left\{ -c_0 \min \left( 2^\ell \frac{t \ell}{b}, \frac{t^2 \ell}{b^2} \right) \right\}
\]

We get the same bound for the 3rd quantity.

\[\Box\]

**Part g), final result**

**Proof.** We decompose \( \frac{1}{\ell} c_0(\varepsilon)t = \sum_{\ell=0}^L t_\ell \), and we will specify the \( t_\ell \) later. Then by e) and f) we get [note that we upper bound the smaller of the 3 inequalities in part f by the other two]:

\[
\mathbb{P} \left[ \|X\|_{op} \geq t \right] \leq 3 \sum_{\ell=0}^L \left( \frac{c_1 n}{2^\ell} \right)^{2 \cdot 2^\ell} \exp \left\{ -c_0 \min \left( 2^\ell \frac{t \ell}{b}, \frac{t^2 \ell}{b^2} \right) \right\}
\]

Next, for some \( s > 0 \), we choose \( t_\ell = b \cdot s \cdot 2^\ell / 2 \). Then for \( s \) large enough, we see that in the \( \min(\cdot) \) expression is attained by the linear term and with value equal to \( s \cdot 2^\ell \). Let us next also note that with the given specification on \( t_\ell \), we get:

\[
\sum_{\ell=0}^L t_\ell = b \cdot s \sum_{\ell=0}^L 2^\ell / 2 = b \cdot s \frac{2^{L+1} - 1}{\sqrt{2} - 1} \approx \frac{1}{\sqrt{2} - 1} b \cdot s \cdot \sqrt{n}
\]

Combining these elements, there exist constants \( c_0, c_1, c_2 \) (unrelated to all constants defined previously) and \( S > 0 \), such that for all \( s \geq S \):

\[
\mathbb{P} \left[ \|X\|_{op} \geq c_0 \cdot s \cdot b \cdot \sqrt{n} \right] \leq c_1 \exp \left( -c_2 s \sqrt{n} \right)
\]

Recall that \( n = d_1 \lor d_2 \), we see that similar to the Subgaussian case here \( \|X\|_{op} = O_P(b\sqrt{d_1 + d_2}) \), however the tail probability decays slower.

\[\Box\]