EE378B Homework 3 Solution

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1 Problem 1

(a) A Laplacian random variable X with density $p(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. We have an explicit form for its MGF,

$$\mathbb{E}\left[\mathrm{e}^{\lambda X}\right] = \frac{1}{2}\left(\int_{-\infty}^{0} \mathrm{e}^{(\lambda+1)x}\mathrm{d}x + \int_{0}^{+\infty} \mathrm{e}^{(\lambda-1)x}\mathrm{d}x\right) = \begin{cases} \frac{1}{1-\lambda^2}, & |\lambda| < 1, \\ +\infty, & \text{else.} \end{cases} \tag{1}$$

Clearly X is not sub-Gaussian. Note that for all $|\lambda| < 1/\sqrt{2}$ it holds that

$$\frac{1}{1-\lambda^2} \le 1 + 2\lambda^2 \le \mathrm{e}^{2\lambda^2}, \tag{2}$$

we can choose $b = 2$, and X is 2-sub-exponential.

(b) For any point $\boldsymbol{u} \in \mathsf{B}_2^n(0,1)$, we construct an $\boldsymbol{x} \in N^n(L)$ that is close to $\boldsymbol{u}$. In particular, let $\mathsf{sign}(u_i) = \mathsf{sign}(x_i)$ and

$$u_i^2 = \begin{cases} 2^{-l}, & 2^{-l} \le x_i^2 < 2^{-l+1} \text{ for some } 0 \le l \le L, \\ 0, & x_i^2 < 2^{-L}. \end{cases} \tag{3}$$

By this construction it clearly holds that $u_i^2 \in S_L, \forall i = 1, 2, \cdots, n$ and $\|\boldsymbol{u}\|_2 \le \|\boldsymbol{x}\|_2 \le 1$, and thus $\boldsymbol{u} \in N^n(L)$. We notice that

$$(x_i - u_i)^2 \le \begin{cases} \left(1 - \frac{1}{\sqrt{2}}\right)^2 x_i^2, & x_i^2 \ge 2^{-L}, \\ 2^{-L}, & x_i^2 \le 2^{-L}, \end{cases} \le \left(1 - \frac{1}{\sqrt{2}}\right)^2 x_i^2 + 2^{-L}, \tag{4}$$

which further suggests,

$$\|\boldsymbol{x} - \boldsymbol{u}\|_2^2 \le \left(1 - \frac{1}{\sqrt{2}}\right)^2 \|\boldsymbol{x}\|_2^2 + n2^{-\log_2 n - c_0} \le \left(1 - \frac{1}{\sqrt{2}}\right)^2 + 2^{-c_0}. \tag{5}$$

For large enough c_0, we can set $\epsilon_0 := \sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + 2^{-c_0}} < \frac{1}{2}$. $N^n(L)$ is then an ϵ_0-net of the unit ball. Finally we use the ϵ_0-net to provide an upper bound for $\|\boldsymbol{X}\|_{\mathrm{op}}$. Simply notice that for any $\boldsymbol{x} \in \mathsf{B}_2^{d_1}(0,1), \boldsymbol{y} \in \mathsf{B}_2^{d_2}(0,1)$, we can find $\tilde{\boldsymbol{x}} \in N_{d_1}(L), \tilde{\boldsymbol{y}} \in N_{d_2}(L)$ such that

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 \le \epsilon_0, \qquad \|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_2 \le \epsilon_0, \tag{6}$$

and thus

$$\begin{aligned} |\langle \boldsymbol{x}, \boldsymbol{X}\boldsymbol{y}\rangle| &\le |\langle \tilde{\boldsymbol{x}}, \boldsymbol{X}\tilde{\boldsymbol{y}}\rangle| + |\langle \boldsymbol{x} - \tilde{\boldsymbol{x}}, \boldsymbol{X}\tilde{\boldsymbol{y}}\rangle| + |\langle \boldsymbol{x}, \boldsymbol{X}\left(\boldsymbol{y} - \tilde{\boldsymbol{y}}\right)\rangle| \\ &\le |\langle \tilde{\boldsymbol{x}}, \boldsymbol{X}\tilde{\boldsymbol{y}}\rangle| + 2\epsilon_0\|\boldsymbol{X}\|_{\mathrm{op}}. \end{aligned} \tag{7}$$

Taking maximum over both sides yields

$$\|\boldsymbol{X}\|_{\mathrm{op}} = \max_{\tilde{\boldsymbol{x}} \in N_{d_1}(L), \tilde{\boldsymbol{y}} \in N_{d_2}(L)} |\langle \boldsymbol{x}, \boldsymbol{X}\boldsymbol{y}\rangle| \le \max_{\boldsymbol{u} \in N_{d_1}(L)} \max_{\boldsymbol{v} \in N_{d_2}(L)} |\langle \boldsymbol{u}, \boldsymbol{X}\boldsymbol{v}\rangle| + 2\epsilon_0\|\boldsymbol{X}\|_{\mathrm{op}}. \tag{8}$$

Hence

$$\{\|\boldsymbol{X}\|_{\text{op}} \geq t\} \subset \left\{ |\langle \boldsymbol{u}, \boldsymbol{X}\boldsymbol{v}\rangle| \geq \frac{t}{1-2\epsilon_0} \right\}, \tag{9}$$

and taking $C(\epsilon_0) = (1-2\epsilon_0)^{-1}$ concludes the proof.

(c) Denote by $\mathcal{B}_k^n$ the collection of all subsets of $[n]$ of cardinality k. Let $\mathcal{C}_k^n := \{\star, -, +\}^k \times \mathcal{B}_k^n$. We claim that there exists two surjections such that

$$\begin{aligned} \varphi_{=l}^n :& \mathcal{C}_{2^l}^n \to N_{=l}^n, & (10)\\ \varphi_{<l}^n :& \prod_{k=0}^{l-1} \mathcal{C}_{2^k}^n \to N_{<l}^n. & (11) \end{aligned}$$

Therefore $|N_{=l}^n| \leq |\mathcal{C}_{2^l}^n|$ and $|N_{<l}^n| \leq \left|\prod_{k=0}^{l-1} \mathcal{C}_{2^k}^n\right|$. While for each $k \leq n$ it holds that

$$|\mathcal{C}_k^n| \leq \binom{n}{k} 3^k \leq \left(\frac{3\mathrm{e}n}{k}\right)^k. \tag{12}$$

Thus it follows that

$$\begin{aligned} |N_{=l}^n| &\leq \left(\frac{3\mathrm{e}n}{2^l}\right)^{2^l}; & (13)\\ |N_{<l}^n| &\leq \prod_{k=0}^{l-1} \left(\frac{3\mathrm{e}n}{2^k}\right)^{2^k} \\ &\leq \exp\left\{ \sum_{k=0}^{l-1} 2^k \log(3\mathrm{e}n) - \sum_{k=0}^{l-1} k 2^k \log 2 \right\} \\ &= \exp\left\{ \left(2^l - 1\right) \log(3\mathrm{e}n) - \left(l2^l - 2(2^l-1)\right) \log 2 \right\} \\ &= \exp\left\{ \left(2^l - 1\right) \log(12\mathrm{e}n) - l2^l \log 2 \right\} \\ &\leq \left(\frac{12\mathrm{e}n}{2^l}\right)^{2^l}. & (14) \end{aligned}$$

Therefore we prove the result for $c_1 = 12\mathrm{e}$. It is only left for us to construct $\varphi_{=l}^n$ and $\varphi_{<l}^n$.

(i) For any $A_{2^l}^n \times B_{2^l}^n \in \mathcal{C}_{2^l}^n$, we define

$$\varphi_{=l}^n(A_{2^l}^n \times B_{2^l}^n)_i = \begin{cases} 0, & \text{if } i \notin B_{2^l}^n \text{ or } i \in B_{2^l}^n \text{ and is the } j\text{-th largest, but } \left(A_{2^l}^n\right)_j = \star; \\ -2^{-l/2}, & \text{if } i \in B_{2^l}^n \text{ and is the } j\text{-th largest, } \left(A_{2^l}^n\right)_j = -; \\ 2^{-l/2}, & \text{if } i \in B_{2^l}^n \text{ and is the } j\text{-th largest, } \left(A_{2^l}^n\right)_j = +. \end{cases} \tag{15}$$

This is clearly a mapping to $N_{=l}^n$, and since for any element in $N_{=l}^n$, there are at most 2^l non-zero coordinates, we have $\varphi_{=l}^n$ is surjective.

(ii) For any $\prod_{k=0}^{l-1} A_{2^l}^k \times B_{2^l}^k \in \prod_{k=0}^{l-1} \mathcal{C}_{2^k}^n$, let

$$\tilde{\varphi}_{<l}^n \left(\prod_{k=0}^{l-1} A_{2^l}^k \times B_{2^l}^k \right)_i = \begin{cases} \text{The only nonzero element among } \varphi_{=k}^n(A_{2^k}^n \times B_{2^k}^n) \text{ for } k = 0, 1, \cdots, l-1; \\ 0, \qquad \text{otherwise,} \end{cases} \tag{16}$$

and

$$\varphi^n_{<l}\left(\prod_{k=0}^{l-1} A^k_{2^l}\times B^k_{2^l}\right) = \begin{cases}\tilde{\varphi}^n_{<l}\left(\prod_{k=0}^{l-1} A^k_{2^l}\times B^k_{2^l}\right), & \left\|\tilde{\varphi}^n_{<l}\left(\prod_{k=0}^{l-1} A^k_{2^l}\times B^k_{2^l}\right)\right\|_2 \le 1,\\ \mathbf{0}, & \left\|\tilde{\varphi}^n_{<l}\left(\prod_{k=0}^{l-1} A^k_{2^l}\times B^k_{2^l}\right)\right\|_2 > 1.\end{cases} \tag{17}$$

Similarly, one can verify that $\varphi^n_{<l}$ maps onto $N^n_{<l}$.

Alternative Approach:

Observe that

$$N^n_{=\ell}\cup N^n_{<\ell}\subset\left\{x\in \mathsf{B}_2(0,1): x_i^2\in\left\{0,1,\frac{1}{2},\dots,\frac{1}{2^\ell}\right\},\forall i\in[n]\right\} = N^n_{<(\ell+1)}.$$

We now count $N^n_{<(\ell+1)}$. Let $y_i = 2^\ell x_i^2$, then y_i are nonnegative integers satisfying $y_1+\cdots+y_n\le 2^\ell$. The number of possibilities for such y is upper bounded by the number of integer solutions to this inequality, which is by stars and bars, $\binom{n+2^\ell}{n}=\binom{n+2^\ell}{2^\ell}$. Further, as $\ell\le L = 2^{c_0}n$, we have

$$\binom{n+2^\ell}{2^\ell}\le\binom{(2^{c_0}+1)n}{2^\ell}\le\left(\frac{(2^{c_0}+1)en}{2^\ell}\right)^{2^\ell}.$$

Finally, notice that $x_i=\pm\sqrt{2^{-\ell}y_i}$, and there are at most 2^ℓ nonzero elements in each y, so we get

$$\left|N^n_{<(\ell+1)}\right|\le 2^{2^\ell}\cdot\left(\frac{(2^{c_0}+1)en}{2^\ell}\right)^{2^\ell} = \left(\frac{2e(2^{c_0}+1)n}{2^\ell}\right)^{2^\ell}.$$

Thus the desired bound holds with $c_1 = 2e(2^{c_0}+1)\le 66e < 180$.

(d) Directly following the definition of sub-exponential, we see that

$$\mathbb{E}e^{\lambda\cdot u_i X_{ij} v_j}\le e^{(\lambda u_i v_j)^2 b^2/2},\qquad \forall|\lambda|<(|u_i v_j|b)^{-1}, \tag{18}$$

since X_{ij} is b-sub-exponential. Thus $u_i X_{ij} v_j$ is $|u_i v_j|b$-sub-exponential. Hence by Bernstein inequality,

$$\begin{aligned}\mathbb{P}\left(|\langle \boldsymbol{u},\boldsymbol{X}\boldsymbol{v}\rangle|\ge t\right) &= \mathbb{P}\left(\left|\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} u_i X_{ij} v_j\right|\ge t\right)\\ &\le 2\exp\left\{-c_0\min\left(\frac{t}{\max_{i=1}^{d_1}\max_{j=1}^{d_2}|u_i v_j|b},\frac{t^2}{\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}u_i^2 v_j^2 b^2}\right)\right\}\\ &\le 2\exp\left\{-c_0\min\left(\frac{t}{b\|\boldsymbol{u}\|_\infty\|\boldsymbol{v}\|_\infty},\frac{t^2}{b^2}\right)\right\},\end{aligned} \tag{19}$$

where in the last inequality we use $\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}u_i^2v_j^2b^2 = b^2\sum_{i=1}^{d_1}u_i^2\sum_{j=1}^{d_2}v_j^2 = b^2\|\boldsymbol{u}\|_2^2\|\boldsymbol{v}\|_2^2\le b^2$ since $\boldsymbol{u}\in\mathsf{B}_2^{d_1}(0,1),\boldsymbol{v}\in\mathsf{B}_2^{d_2}(0,1)$.

(e) By the definition of $\pi_{<l}$ and $\pi_{=l}$, we have for any $\boldsymbol{x}\in N^n(L)$ that,

$$\boldsymbol{x} = \pi_{=L}(\boldsymbol{x})+\pi_{<L}(\boldsymbol{x}), \tag{20}$$

$$\pi_{<l}(\boldsymbol{x}) = \sum_{k=0}^{l-1}\pi_{=k}(\boldsymbol{x}). \tag{21}$$

Hence for any $\boldsymbol{u}\in N^{d_1}(L),\boldsymbol{v}\in N^{d_2}(L)$,

$$\langle\boldsymbol{u},\boldsymbol{M}\boldsymbol{v}\rangle = \langle\pi_{=L}(\boldsymbol{u})+\pi_{<L}(\boldsymbol{u}),\boldsymbol{M}\left(\pi_{=L}(\boldsymbol{v})+\pi_{<L}(\boldsymbol{v})\right). \tag{22}$$

Notice that for any $l = 1, 2, \cdots, L$ one has

$$\begin{aligned}
&\langle \pi_{=l}(\boldsymbol{u}) + \pi_{<l}(\boldsymbol{u}), \boldsymbol{M}\left(\pi_{=l}(\boldsymbol{v}) + \pi_{<l}(\boldsymbol{v})\right) \\
&= \langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{M}\pi_{=l}(\boldsymbol{v})\rangle + \langle \pi_{<l}(\boldsymbol{u}), \boldsymbol{M}\pi_{=l}(\boldsymbol{v})\rangle + \langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{M}\pi_{<l}(\boldsymbol{v})\rangle + \langle \pi_{<l}(\boldsymbol{u}), \boldsymbol{M}\pi_{<l}(\boldsymbol{v})\rangle \\
&= \langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{M}\pi_{=l}(\boldsymbol{v})\rangle + \langle \pi_{<l}(\boldsymbol{u}), \boldsymbol{M}\pi_{=l}(\boldsymbol{v})\rangle + \langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{M}\pi_{<l}(\boldsymbol{v})\rangle \\
&\quad + \langle \pi_{=l-1}(\boldsymbol{u}) + \pi_{<l-1}(\boldsymbol{u}), \boldsymbol{M}\left(\pi_{=l-1}(\boldsymbol{v}) + \pi_{<l-1}(\boldsymbol{v})\right)\rangle. \qquad (23)
\end{aligned}$$

We thus can expand $\langle \boldsymbol{u}, \boldsymbol{M}\boldsymbol{v}\rangle$ iteratively using the equation above for $l = L, L-1, \cdots, 1$ and obtain

$$\langle \boldsymbol{u}, \boldsymbol{M}\boldsymbol{v}\rangle = \sum_{l=0}^{L} \langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{M}\pi_{=l}(\boldsymbol{v})\rangle + \sum_{l=0}^{L} \langle \pi_{<l}(\boldsymbol{u}), \boldsymbol{M}\pi_{=l}(\boldsymbol{v})\rangle + \sum_{l=0}^{L} \langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{M}\pi_{<l}(\boldsymbol{v})\rangle. \qquad (24)$$

The proof is done.

(f) We use the results from (c) and (d) and make use of the fact that $\|\pi_{=l}(\boldsymbol{u})\|_\infty \le 2^{-l/2}$. Thus

$$\begin{aligned}
&\mathbb{P}\left(\max_{\boldsymbol{u}\in N_{d_1}(L)} \max_{\boldsymbol{v}\in N_{d_2}(L)} |\langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{X}\pi_{=l}(\boldsymbol{v})\rangle| \ge t_l\right) \\
&\le |N_{=l}^{d_1}||N_{=l}^{d_2}| \cdot 2\exp\left\{-c_0 \min\left(\frac{t_l}{b\cdot 2^{-l/2}\cdot 2^{-l/2}}, \frac{t_l^2}{b^2}\right)\right\} \\
&\le 2\left(\frac{c_1 d_1}{2^l}\right)^{2^l}\left(\frac{c_1 d_2}{2^l}\right)^{2^l} \exp\left\{-c_0\min\left(\frac{t_l 2^l}{b}, \frac{t_l^2}{b^2}\right)\right\} \\
&= 2\left(\frac{c_1^2 d_1 d_2}{2^l}\right)^{2^l} \exp\left\{-c_0\min\left(\frac{t_l 2^l}{b}, \frac{t_l^2}{b^2}\right)\right\}. \qquad (25)
\end{aligned}$$

Similarly we get

$$\mathbb{P}\left(\max_{\boldsymbol{u}\in N_{d_1}(L)} \max_{\boldsymbol{v}\in N_{d_2}(L)} |\langle \pi_{<l}(\boldsymbol{u}), \boldsymbol{X}\pi_{=l}(\boldsymbol{v})\rangle| \ge t_l\right) \le 2\left(\frac{c_1^2 d_1 d_2}{2^l}\right)^{2^l} \exp\left\{-c_0\min\left(\frac{t_l 2^{l/2}}{b}, \frac{t_l^2}{b^2}\right)\right\}, \qquad (26)$$

$$\mathbb{P}\left(\max_{\boldsymbol{u}\in N_{d_1}(L)} \max_{\boldsymbol{v}\in N_{d_2}(L)} |\langle \pi_{=l}(\boldsymbol{u}), \boldsymbol{X}\pi_{<l}(\boldsymbol{v})\rangle| \ge t_l\right) \le 2\left(\frac{c_1^2 d_1 d_2}{2^l}\right)^{2^l} \exp\left\{-c_0\min\left(\frac{t_l 2^{l/2}}{b}, \frac{t_l^2}{b^2}\right)\right\}. \qquad (27)$$

(g) Take $t_l := Cb2^{l/2}$ into (f) for some constant C to be determined. Then by (e),

$$\begin{aligned}
&\mathbb{P}\left(\max_{\boldsymbol{u}\in N_{d_1}(L)} \max_{\boldsymbol{v}\in N_{d_2}(L)} |\langle \boldsymbol{u}, \boldsymbol{X}\boldsymbol{v}\rangle| \ge 3Cb\sum_{l=0}^{L} 2^{l/2}\right) \\
&\le 6\sum_{l=0}^{L}\left(\frac{c_1^2 d_1 d_2}{2^l}\right)^{2^l} \exp\left\{-c_0 2^l \min\left(C, C^2\right)\right\} \\
&\le 6\sum_{l=0}^{L} \exp\left\{2^l\left(\log\left(\frac{c_1^2 d_1 d_2}{2^l}\right) - c_0\min\left(C, C^2\right)\right)\right\}. \qquad (28)
\end{aligned}$$

By taking $C \ge \frac{2\log\left(c_1^2 d_1 d_2\right)}{c_0} \vee 1$, then

$$\log\left(\frac{c_1^2 d_1 d_2}{2^l}\right) - c_0\min\left(C, C^2\right) = \log\left(\frac{c_1^2 d_1 d_2}{2^l}\right) - c_0 C \le -\frac{c_0 C}{2}, \qquad (29)$$

and therefore

$$\sum_{l=0}^{L} \exp\left\{2^l\left(\log\left(\frac{c_1^2 d_1 d_2}{2^l}\right) - c_0\min\left(C, C^2\right)\right)\right\} \le \sum_{l=1}^{+\infty} \exp\left(-\frac{lc_0 C}{2}\right) \le \frac{\exp\left(-\frac{c_0 C}{2}\right)}{1-\exp\left(-\frac{c_0 C}{2}\right)}. \qquad (30)$$

On the other hand take $L = \log_2(d_1 \vee d_2) + c_0$,

$$\sum_{l=0}^{L} 2^{l/2} = \frac{2^{(L+1)/2} - 1}{\sqrt{2} - 1} \geq 3 \cdot 2^{(\log_2 n + c_0 + 1)/2} \geq 3 \cdot 2^{\frac{c_0+1}{2}} \sqrt{d_1 \vee d_2}. \tag{31}$$

Taken collectively, we have

$$\mathbb{P}\left(\max_{\boldsymbol{u} \in N_{d_1}(L)} \max_{\boldsymbol{v} \in N_{d_2}(L)} |\langle \boldsymbol{u}, \boldsymbol{X}\boldsymbol{v} \rangle| \geq 9 \cdot 2^{\frac{c_0+1}{2}} Cb\sqrt{d_1 \vee d_2} \right) \leq \frac{6 \exp\left(-\frac{c_0 C}{2}\right)}{1 - \exp\left(-\frac{c_0 C}{2}\right)}, \tag{32}$$

for all $C \geq \frac{2\log\left(c_1^2 d_1 d_2\right)}{c_0} \vee 1$. Taking into (b) and we finally get

$$\mathbb{P}\left(\|\boldsymbol{X}\|_{\mathrm{op}} \geq C_1 b\sqrt{d_1 \vee d_2 t} \right) \leq C_2 \exp\left(-C_3 t\right), \qquad \forall t \geq C_4 \log\left(d_1 \vee d_2\right), \tag{33}$$

for some universal constants $C_1, C_2, C_3, C_4 > 0$.