EE378B Homework 7 Solution

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1 Part (a)

Direct application of Wielandt-Hoffman inequality gives

$$\sum_{i=1}^{n} \left|\lambda_i\left(\boldsymbol{W}_n/\sqrt{n}\right) - \lambda_i\left(\boldsymbol{W}_n^K/\sqrt{n}\right)\right|^2 \le \frac{1}{n}\left\|\boldsymbol{W}_n - \boldsymbol{W}_n^K\right\|_F^2 = \frac{1}{n}\sum_{i\neq j}\xi_{ij}, \tag{1}$$

where ξ_{ij} are i.i.d. random variables having the same distribution as $W_{12}^2 \mathbf{1}_{|W_{12}|>K}$. Therefore by SLLN with probability 1,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left|\lambda_i\left(\boldsymbol{W}_n/\sqrt{n}\right) - \lambda_i\left(\boldsymbol{W}_n^K/\sqrt{n}\right)\right|^2 &\le \limsup_{n\to\infty} \frac{1}{n^2}\sum_{i\neq j}\xi_{ij} \\
&= \lim_{n\to\infty} \frac{n-1}{n}\cdot\frac{1}{n(n-1)}\sum_{i\neq j}\xi_{ij} \\
&= \mathbb{E}\left[W_{12}^2\mathbf{1}_{|W_{12}|>K}\right].
\end{aligned} \tag{2}$$

2 Part (b)

For simplicity we write

$$\frac{1}{n}\sum_{i=1}^{n} \left|\lambda_i\left(\boldsymbol{W}_n/\sqrt{n}\right) - \lambda_i\left(\boldsymbol{W}_n^K/\sqrt{n}\right)\right|^2 = \Delta_{n,K}. \tag{3}$$

As a consequence of part (a) we know $\limsup_{n\to\infty} \Delta_{n,K} \le \Delta_K$. By Markov's inequality

$$\frac{1}{n}\#\left\{i : \left|\lambda_i\left(\boldsymbol{W}_n/\sqrt{n}\right) - \lambda_i\left(\boldsymbol{W}_n^K/\sqrt{n}\right)\right| \ge \epsilon\right\} \le \frac{\Delta_{n,K}}{\epsilon^2}, \tag{4}$$

which allows us to derive that

$$\begin{aligned}
F_n^K(x-\epsilon) - F_n(x) &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{1}_{\lambda_i(\boldsymbol{W}_n^K/\sqrt{n})\le x-\epsilon} - \mathbb{1}_{\lambda_i(\boldsymbol{W}_n/\sqrt{n})\le x}\right) \\
&\le \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\lambda_i(\boldsymbol{W}_n^K/\sqrt{n})\le x-\epsilon, \lambda_i(\boldsymbol{W}_n/\sqrt{n})> x} \\
&\le \frac{1}{n}\#\left\{i : \left|\lambda_i\left(\boldsymbol{W}_n/\sqrt{n}\right) - \lambda_i\left(\boldsymbol{W}_n^K/\sqrt{n}\right)\right|\right\} \\
&\le \frac{\Delta_{n,K}}{\epsilon^2}.
\end{aligned} \tag{5}$$

Similarly we bound $F_n(x) - F_n^K(x+\epsilon) \le \frac{\Delta_{n,K}}{\epsilon^2}$. For the above two inequalities, take lim sup on both sides and rearrange terms, which gives the desired result

$$F^K(x-\epsilon) - \frac{\Delta_K}{\epsilon^2} \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F^K(x+\epsilon) + \frac{\Delta_K}{\epsilon^2}. \tag{6}$$

Finally, if $F^K(x) \to F(x)$ for every $x \in \mathbb{R}$ as $K \to \infty$, using the fact that $\Delta_K \to 0$ since W_{12} has finite second moment, we obtain by taking $K \to \infty$ that

$$F(x-\epsilon) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x+\epsilon). \tag{7}$$

Whenever F is continuous at x, by taking $\epsilon \to 0$ it is concluded that $\lim_{n\to\infty} F_n(x) = F(x)$.

3 Part (c)

In fact, it suffices to show that for all K with $\beta_K = \mathrm{Var}\left(W_{12}\mathbb{1}_{|W_{12}|\le K}\right)$,

$$F_n^K(x) \to F^K(x) := \frac{1}{2\beta_K \pi}\int_{-\infty}^{x} \sqrt{4\beta_K - t^2}\mathbb{1}_{|t|\le 2\sqrt{\beta_K}}\mathrm{d}t, \tag{8}$$

i.e., the semi-circular law with variance β_K. Then all conditions hold since $\beta_K \to 1$ as $K \to \infty$ and $F^K(x) \to F(x)$ pointwise with $F(x)$ being the semi-circular law CDF with variance 1.

Let $\tilde{\boldsymbol{W}}_n^K = \boldsymbol{W}_n^K + \alpha_K \mathbf{1}\mathbf{1}^\top$ where $\alpha_K = -\mathbb{E}[W_{12}\mathbb{1}_{|W_{12}|\le K}]$. Then $\mathbb{E}\tilde{\boldsymbol{W}}_n^K = \mathbf{0}$ and we can apply results that have been shown in class that

$$\tilde{F}_n^K(x) \to F^K(x), \tag{9}$$

where $\tilde{F}_n^K(x)$ is the empirical CDF of eigenvalues of $\tilde{\boldsymbol{W}}_n^K/\sqrt{n}$. The proof will be concluded once we can establish

$$\left|\tilde{F}_n^K(x) - F_n^K(x)\right| \to 0 \tag{10}$$

for any fixed K and $x \in \mathbb{R}$. By the generalized Weyl's inequality, we're able to get

$$\lambda_i(\tilde{\boldsymbol{W}}_n^K/\sqrt{n}) \ge \lambda_{i+1}(\boldsymbol{W}_n^K/\sqrt{n}) + \lambda_{n-1}(\alpha_K\mathbf{1}\mathbf{1}^\top/\sqrt{n}), \qquad i = 1, 2, \cdots, n-1; \tag{11}$$

$$\lambda_i(\tilde{\boldsymbol{W}}_n^K/\sqrt{n}) \le \lambda_{i-1}(\boldsymbol{W}_n^K/\sqrt{n}) + \lambda_2(\alpha_K\mathbf{1}\mathbf{1}^\top/\sqrt{n}), \qquad i = 2, 3, \cdots, n. \tag{12}$$

While $\alpha_K\mathbf{1}\mathbf{1}^\top/\sqrt{n}$ is only a rank-1 matrix, it follows when $n \ge 3$ that

$$\lambda_2(\alpha_K\mathbf{1}\mathbf{1}^\top/\sqrt{n}) = \lambda_{n-1}(\alpha_K\mathbf{1}\mathbf{1}^\top/\sqrt{n}) = 0, \tag{13}$$

which gives us

$$\lambda_i(\tilde{\boldsymbol{W}}_n^K/\sqrt{n}) \ge \lambda_{i+1}(\boldsymbol{W}_n^K/\sqrt{n}), \qquad i = 1, 2, \cdots, n-1; \tag{14}$$

$$\lambda_i(\tilde{\boldsymbol{W}}_n^K/\sqrt{n}) \le \lambda_{i-1}(\boldsymbol{W}_n^K/\sqrt{n}), \qquad i = 2, 3, \cdots, n. \tag{15}$$

By the first bound, one can deduce that

$$\begin{aligned}\tilde{F}_n^K(x) - F_n^K(x) &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{1}_{\lambda_i(\tilde{\boldsymbol{W}}_n^K/\sqrt{n})\le x} - \mathbb{1}_{\lambda_i(\boldsymbol{W}_n^K/\sqrt{n})\le x}\right)\\ &\le \frac{1}{n}\sum_{i=1}^{n-1}\left(\mathbb{1}_{\lambda_i(\tilde{\boldsymbol{W}}_n^K/\sqrt{n})\le x} - \mathbb{1}_{\lambda_{i+1}(\boldsymbol{W}_n^K/\sqrt{n})\le x}\right) + \frac{1}{n}\\ &\le \frac{1}{n}.\end{aligned} \tag{16}$$

The other side is similar, and therefore $\left|\tilde{F}_n^K(x) - F_n^K(x)\right| \le 1/n \to 0$. The proof is done.

4 Part (d)

We can generate uniformly distributed (in the sense of Haar's measure) orthogonal matrices $\boldsymbol{U} \in \mathbb{R}^{n\times k}$ by first generate an n by k matrix with i.i.d. standard Gaussian entries and perform Gram-Schmidt orthogonalization. Since the distribution of Gaussian matrix is invariant under orthogonal transformation, the distribution of resulting orthogonal matrix $\boldsymbol{U}$ is also invariant, and thus is uniformly distributed under Haar's measure.

(i) The plots are shown in Figure 1.

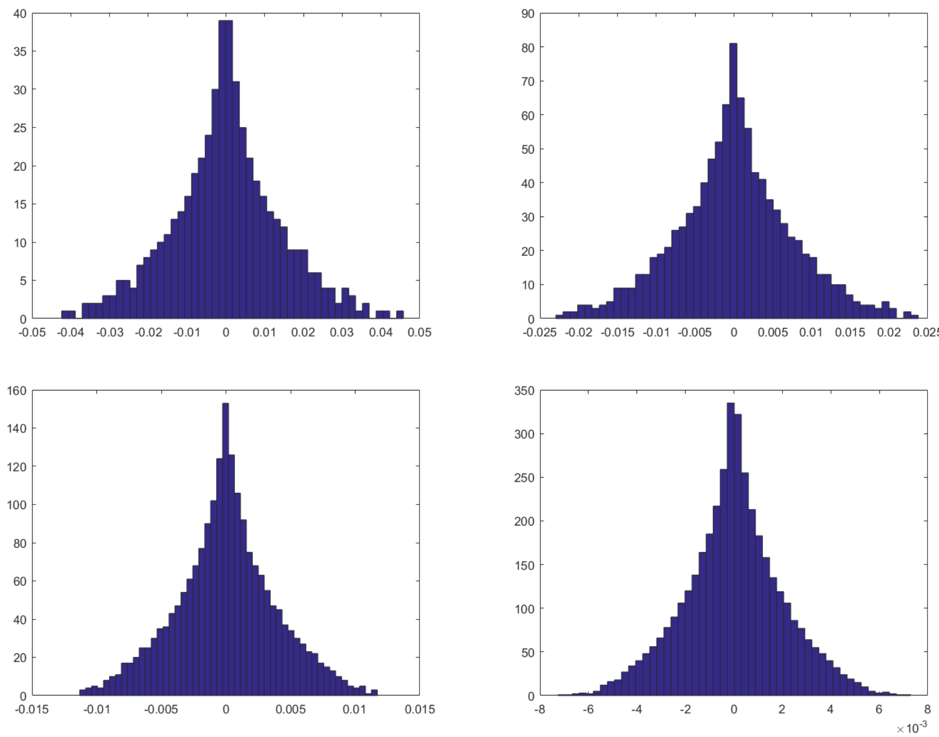

Figure 1: Histograms of eigenvalues of $\boldsymbol{Y}$ when $n = 500$ (left top), $n = 1000$ (right top), $n = 2000$ (left down), $n = 4000$ (right down).

(ii) The result is different from Winger's semicircle's law whereas the histogram should look like a semicircle.

(iii) We can't apply since the entries Y_{ij} are correlated.

Remark from class: What happens for a larger choice of constant? Does it recover the semicircle law?