Analysis of Maximum-Size Matching Scheduling Algorithms (MSM) in input-queue switches under uniform traffic

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1 Introduction

In the design of input-queue switches, scheduling algorithms play an important role. It has been shown that the Maximum Weight Matching Algorithm (MWM) gives 100% throughput, but it's difficult to implement. (It has running time complexity $\mathrm{O}(N^3 \log N)$. Another scheduling algorithm is the Maximum Size Matching (MSM) Algorithm which maximizes the instantaneous throughput with less running time complexity, $\mathrm{O}(N^{2.5})$. MSM is known to be unstable under non-uniform traffic pattern[3]. Although there have been results on the stability of some specific MSM for scheduling traffic [8], the question of how it performs under uniform traffic remains open. Simulations suggest the stability of MSM algorithms under uniform traffic but there have been no analytical results proving the same. Our goal is to find a rigorous analytical argument showing this or finding a counterexample.

2 Background and Motivation

In the theory of dynamical systems the method of Lyapunov functions is a way to show the stability of solutions of differential equations. This method has been developed and applied to Markov Chains for showing the stability (Foster's criterion), as it has been done for MWM [3] or LPF [8]

In an $N \times N$ switch if the number of packets waiting at input i for output j at time n, is denoted by $q_{ij}(n)$, It is not hard to show that Foster's criterion satisfies for the quardratic Lyapunov function $L_1(n) = \sum_{i,j} q_{ij}(n)^2$ when the random policy is applied (i.e. At any time-slot one of the possible $N!$ matchings is chosen at random for scheduling). This function also can be used to show the stability of MWM [3]. But it fails for MSM and one main reason is that weight of queues are not considered in MSM but the function depends highly on the weight of queues.

Mekkitikul and McKeown's work in showing the stability of LPF [8] suggests that more general quadratic Lyapunov functions might help to overcome these difficulties.

3 Problem Statement

Our goal is to find a suitable quadratic Lyapunov function to show the stability of the switch under uniform traffic when MSM is applied. (i.e. at every time slot the matching with the maximum size is chosen for scheduling)

4 Current Results

Consider the following Lyapunov function:

$$L(n) = \sum_{i,j} q_{ij}(n)^2 + \frac{2}{3} \sum_{i,j,r,s} f_{ijrs} q_{ij}(n) q_{rs}(n) - \frac{2}{3} \sum_{i,j,r,s} g_{ijrs} q_{ij}(n) q_{rs}(n)$$

$$f_{ijrs} = \begin{cases} 1 & \text{if edges } \{i,j\} \text{ and } \{r,s\} \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

and $g_{ijrs} = 1 - f_{ijrs}$.

For simplicity, let's write this quadratic form in matrix format: $L(n) = Q(n)^t PQ(n)$ with P the appropriate $N^2 \times N^2$ matrix and $Q(n)$ is a $N^2 \times 1$ vector with entries $q_{ij}(n)$.

Theorem 1 (Foster's Criteria) There exists an $\epsilon > 0$ for which we have[1]:

$$\mathbb{E}(L(n+1) - L(n)|Q(n)) \leq -\epsilon \left(\sum_{ij} q_{ij}(n) \right) + c(N) \qquad (1)$$

where $c(n)$ is a constant depending only on N.

PROOF: First of all for all i, j, n, let $\pi_{ij}(n)$ be the random variable defined as:

$$\pi_{ij}(n) = \begin{cases} 1 & \text{at } n^{th} \text{ step, MSM uses edge } \{i,j\} \\ 0 & \text{otherwise} \end{cases}$$

If the arrival and departure vectors are denoted by $A(n)$ and $D(n)$ respectively then for the drift of $L(n)$ we have:

$$\begin{aligned} \mathbb{E}\Big(L(n+1) - L(n)|Q(n)\Big) &= \mathbb{E}\Big(Q(n+1)^t PQ(n+1) - Q(n)^t PQ(n)|Q(n)\Big) \\ &= \mathbb{E}\Big(2Q(n)^t PA(n) + A(n)^t PA(n) + D(n)^t PD(n) \\ &- 2D(n)^t PA(n) - 2Q(n)^t PD(n)|Q(n)\Big) \\ &= \mathbb{E}\Big(2Q(n)^t PA(n) - 2Q(n)^t PD(n)|Q(n)\Big) \\ &+ \mathbb{E}\Big(A(n)^t PA(n) + D(n)^t PD(n) - 2D(n)^t PA(n)|Q(n)\Big) \end{aligned}$$

[1] This is the same as the Foster's Criteria because if at least one of $q_{ij}(n)$ becomes very large then right hand side of (1) will become negative.

The second term above (i.e. $\mathbb{E}(A(n)^t PA(n)+D(n)^t PD(n)-2D(n)^t PA(n)|Q(n))$) is bounded by a constant called $c(N)$, because the vectors $A(n)$ and $D(n)$ have entries at most equal to one and P is a fixed matrix for all n[2]. So we need to restric ourself to the first term which we call it ★.

Since arrivals are independent of queue sizes we have:

$$\begin{aligned}
\mathbb{E}(2Q(n)^t PA(n)|Q(n)) &= 2\left(\sum_{ij} q_{ij}(n)\right)\left(\frac{\text{sum of any row or column of P}}{N}\right) \\
&= 2\left(\sum_{ij} q_{ij}(n)\right)\left(1+\frac{2(N-1)}{3}-\frac{(N-1)^2}{3}\right)\lambda \\
&= 2\left(\sum_{ij} q_{ij}(n)\right)\left(\frac{4-N}{3}\right)\lambda
\end{aligned}$$

Let p_{ijrs} be the entry in row ij and column rs of the matrix P then we can write:

$$\begin{aligned}
\mathbb{E}(2Q(n)^t PD(n)|Q(n)) &= 2\sum_{ij} q_{ij}(n)\sum_{rs} p_{ijrs}\mathbb{E}(d_{rs}(n)) \\
&= 2\sum_{ij} q_{ij}(n)\sum_{rs} p_{ijrs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n))
\end{aligned}$$

Now ★ can be written as follows: (assuming $\mathbb{E}(A_{ij}(n))=\lambda$ for all i,j,n.)

$$2\sum_{ij} q_{ij}(n)\left(\left(\frac{4-N}{3}\right)\lambda-\sum_{rs} p_{ijrs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n))\right)$$

Let $\epsilon = 2(\frac{1}{N}-\lambda)$ which by admissibility of the arrivals is positive. Now to show (1) we prove for all ij:

$$q_{ij}(n)\left(\sum_{rs} p_{ijrs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n))\right) \;\geq\; q_{ij}(n)\left(\frac{4-N}{3}\right)$$

[2]If at time slot n the arrival to q_{ij} is denoted by $A_{ij}(n)$ we assume $\mathbb{E}(A_{ij}^2(n))<\infty$.

Note that this holds when $q_{ij}(n) = 0$ and by symmetry of all formulas we just need to show this for $i = j = 1$, so from now on assume $q_{11}(n) > 0$. And our goal is to show:

$$\sum_{rs} p_{11rs}\mathbb{E}(\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}|Q(n)) \;\geq\; \frac{4-N}{3}$$

For simplicity let's assume: $Y_{11} = \sum_{rs} p_{11rs}\pi_{rs}(n)\mathbf{1}_{\{q_{rs}(n)>0\}}$. Let's put all the edges of the graph other that q_{11} in two different groups:

- $\mathbf{A} = \{$all edges that are adjacent to $q_{11}\}$ (i.e. all q_{ij} such that $f_{11ij} = 1$)
- $\mathbf{B} = \{$all edges that are non-adjacent to $q_{11}\}$ (i.e. all q_{ij} such that $g_{11ij} = 1$)

Now consider all types of possible maximum size matchings, since $q_{11}(n) > 0$ there are only three possibilities:

- **a)** q_{11} and k edges from $\mathbf{B}$ for $0 \leq k \leq N-1$
- **b)** *Two* edges from the set $\mathbf{A}$ and k edges from $\mathbf{B}$ for $0 \leq k \leq N-2$
- **c)** *One* edge from the set $\mathbf{A}$ and k edges from $\mathbf{B}$ for $0 \leq k \leq N-2$

For matchings of type **a** we have $Y_{11} = 1 - \frac{k}{3} \geq 1 - \frac{N-1}{3} = \frac{4-N}{3}$ and for matchings of type **b** we have $Y_{11} = \frac{2}{3} - \frac{k}{3} \geq \frac{2}{3} - \frac{N-2}{3} = \frac{4-N}{3}$ hence in both cases the inequality $Y_{11} \geq \frac{4-N}{3}$ holds. The only case we might worry is type **c** matchings but in that case also things work out in a fine way, because if a type **c** is a maximum size matching, choosing q_{11} instead of its edge from $\mathbf{A}$ we obtain a type **a** maximum size matching as well and since MSM algorithm always chooses one matching at random among the set of all possible maximum size matchings with equal probability it can choose a type **a** matching.

Now the question is, a set of k edges from $\mathbf{B}$ can be used in how many type **a** and how many type **c** matchings at the same time. And the answer is at most $N - k + 1$, because the vertices already used by $\mathbf{B}$ edges cannot be used for $\mathbf{A}$ edges and also all the used $\mathbf{A}$ edges have to use only one of the endpoints of q_{11}, otherwise we would get a matching of size $k+2$ which contradicts the fact that original type **c** matching with $k+1$ edges is maximum size. Using all of these facts we get:

$$\mathbb{E}(Y_{11}|Q(n)) \geq \frac{1-\frac{k}{3}+(N-k-1)(\frac{1}{3}-\frac{k}{3})}{N-k} \geq \frac{4-N}{3}$$

■

Now we have to check whether $L(n)$ is non-negative function. We can show for $N = 2, 3, 4$ this is the case and it proves all we need to show the stability of MSM under uniform arrival traffic for $2 \times 2, 3 \times 3$ and 4×4 input-queued switches.

Theorem 2 $L(n)$ is non-negative when $N = 2, 3, 4$.

PROOF: Let $R(N, x, y, z)$ be the $N^2 \times N^2$ matrix whose entries r_{ijrs} are as follows:

$$r_{ijrs} = \begin{cases} x & \text{if edges } f_{ijrs} = 1 \\ y & \text{if edges } g_{ijrs} = 1 \\ z & \text{if } \{i, j\} = \{r, s\} \end{cases}$$

For all N we have $P = R(N, \frac{1}{3}, -\frac{1}{3}, 1)$. When $N = 2$, let $R_1 = R(2, \frac{1}{3}, 0, \frac{2}{3})$ and $R_2 = R(2, 0, -\frac{1}{3}, \frac{1}{3})$. Using matlab we check that R_2 is positive definite matrix and since entries of R_1 are non-negative then by the fact that queue sizes are not negative numbers then

$$Q(n)^t P Q(n) = Q(n)^t R_1 Q(n) + Q(n)^t R_2 Q(n) \geq 0$$

Similarly for $N = 3$ $R_2 = R(3, \frac{1}{6}, -\frac{1}{3}, 1)$ is positive definite and $R_1 = R(3, \frac{1}{6}, 0, 0)$ has non-negative entries. And for $N = 4$ the matrix P is itself non-negative definite.

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At the end, we show that in a 2×2 switch, MSM algorithm gives 100% throughput for the more general arrival traffic matrix

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$

where $\lambda_1 + \lambda_2 < 1$. To show this, rewriting the constraint $\mathbb{E}(Y_{11}|Q(n)) \geq \frac{4-N}{3}$ for this arrival traffic matrix we get:

$$\begin{aligned} Y_{11} &\geq \lambda_1 + \frac{2}{3}\lambda_2 - \frac{1}{3}\lambda_1 \\ &= \frac{2}{3}(\lambda_1 + \lambda_2) \end{aligned}$$

which holds for $\lambda_1 + \lambda_2 < 1$.

5 Remaining Work

We know our function $L(n)$ is not positive for $N > 5$ but it depends on some constants which, changing them can make the function positive but could make the drift no longer negative. So we need to modify the function in order to show the stability for all N. Also the stability region of $N \times N$ switch under more generalized input traffic can be studied.

6 Final Results

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