Exploiting Statistical Dependence for Compression

- Joint entropy
- Statistical dependence among color components
- Conditional entropy and bit-rate for lossless coding of sources with memory
- Image conditional entropy measurements
- Cross entropy
- Run-length coding
- Facsimile compression standards

Joint entropy

- Consider random vectors (with finite-alphabet components)

$$\mathbf{X} = \left(X_0, X_1, \ldots, X_{m-1} \right)$$

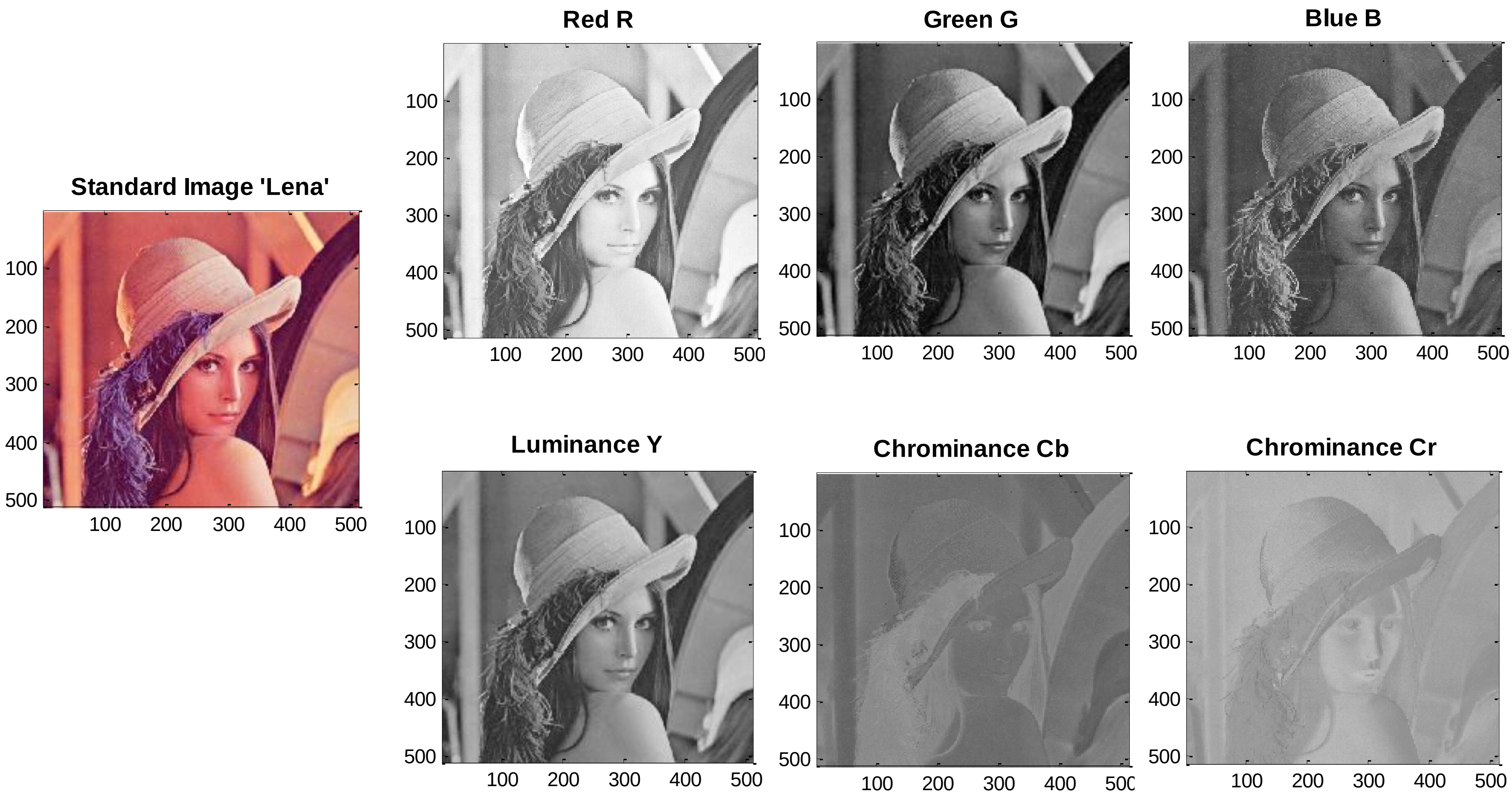
- Entropy $$\boxed{H(\mathbf{X}) = E\left[-\log_2 f_{\mathbf{X}}\left(\mathbf{X}\right) \right] = E\left[h_{\mathbf{X}}\left(\mathbf{X}\right) \right]}$$

- Long-hand $H(\mathbf{X}) \equiv H(X_0, X_1, \ldots, X_{m-1})$

$$= -\sum_{x_0} \sum_{x_1} \cdots \sum_{x_{m-1}} f_{\mathbf{X}}\left(x_0, x_1, \ldots, x_{m-1}\right) \log_2 f_{\mathbf{X}}\left(x_0, x_1, \ldots, x_{m-1}\right)$$

- <u>Shannon's Noiseless Source Coding Theorem</u>: Consider a "vector source" generating i.i.d. random vectors $\mathbf{X}$. Joint entropy $H(\mathbf{X})$ is achievable lower bound for bit-rate for encoding $\mathbf{X}$.

Color components



Joint entropy and statistical dependence

- Theorem

$$H(X_0, X_1, X_2, ..., X_{m-1}) \le H(X_0) + H(X_1) + ... + H(X_{m-1})$$

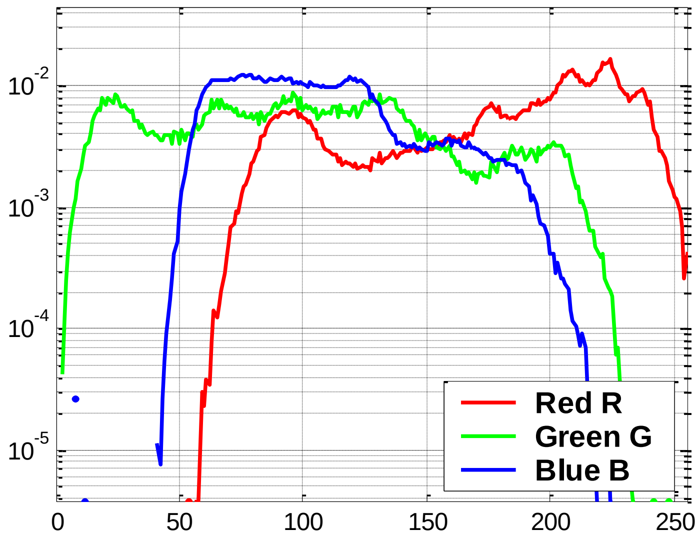
Equality for statistical independence of $X_0, X_1, X_2, ..., X_{m-1}$

- Exploiting statistical dependence lowers bit-rate
- Statistically independent components can be coded separately without loss in efficiency

First order statistics of color components

- Histograms (relative number of occurrences) used in lieu of PMFs

Standard Lena Image - PMF of RGB Pixel Values

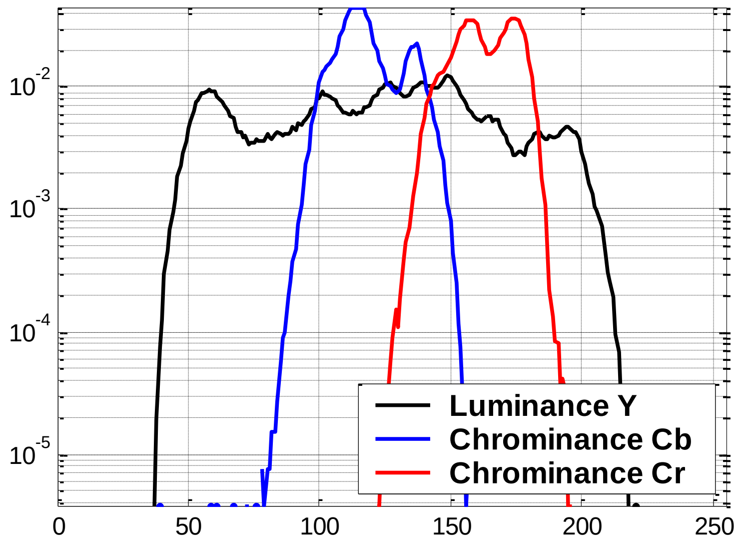


$H(R)$=7.25 bpp

$H(G)$=7.59 bpp

$H(B)$=6.97 bpp

Standard Lena Image - PMF of YCbCr Pixel Values



$H(Y)$=7.23 bpp

$H(Cb)$=5.47 bpp

$H(Cr)$=5.42 bpp

- Image 'Lena', 512 x 512 pixels, 8 bits per color components

Statistical dependence among color components

- Image: 'Lena', 512 x 512 pixels, 8 bits per color component

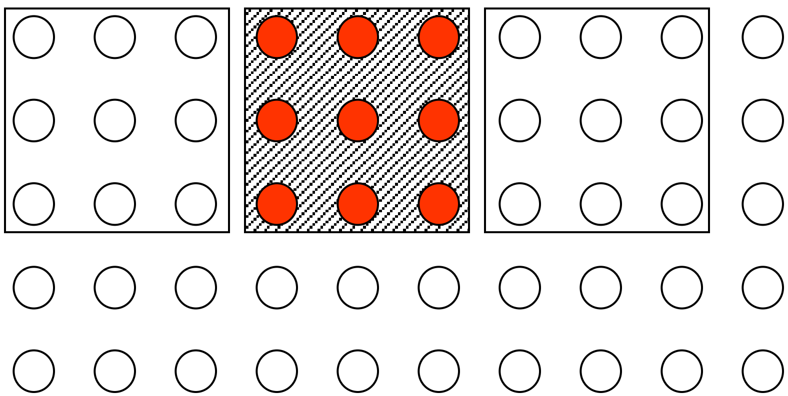
Fixed length	3 x 8 = 24 bpp
$H(Y,Cb,Cr)$	15.01 bpp
$H(Y)+H(Cb)+H(Cr)$	18.12 bpp
ΔH	3.11 bpp

Fixed length	3 x 8 = 24 bpp
$H(R,G,B)$	16.84 bpp
$H(R)+H(G)+H(B)$	21.82 bpp
ΔH	4.98 bpp

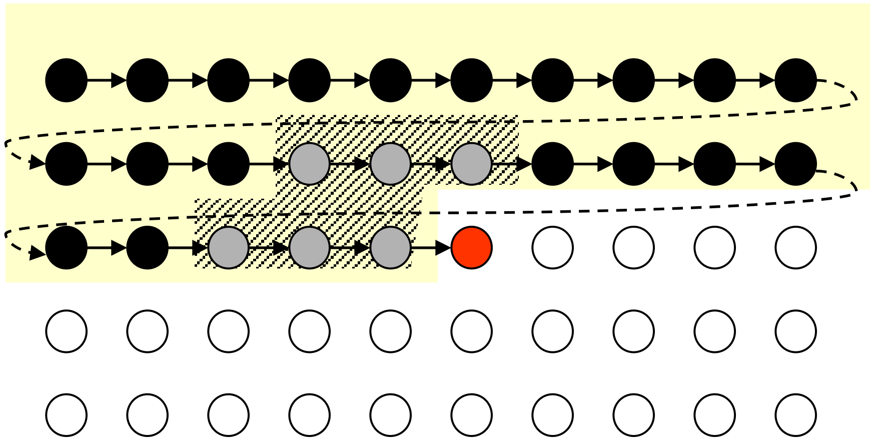
- Statistical dependence among R, G, B is stronger.
- Caveat: if joint sources Y, Cb, Cr or R,G,B are not treated as i.i.d., the possible gain by joint coding may be much smaller.

Exploiting Statistical Dependence Among Neighboring Pixels

- Break image into blocks
- Interpret each block as vector
- Block-by-block entropy coding
- Neglects dependencies across block boundaries



- Sliding window across image
- Interpret image as source with memory
- Perform conditional entropy coding



Markov random processes

- Consider discrete random process $\{X_n\}$ with memory
- Stationarity

$$f_{\mathbf{X}_{i:i+m}} = f_{\mathbf{X}_{0:m}} \quad \text{for all } i, m \in \mathbb{Z}, m \geq 0$$

($m+1$)-dimensional random vector, Components $0 \ldots m$ of infinite-dimensional random vector

- Random process with finite memory (“Markov-p process”)

$$f_{X_m|\mathbf{X}_{0:(m-1)}} = f_{X_m|\mathbf{X}_{(m-p)(m-1)}}$$

- Order-p conditional PMF describes stationary Markov process fully

Conditional entropy

- Consider two finite-alphabet r.v. X and Y

$$H(X|Y) = E\left[-\log_2 f_{X|Y}(x,y)\right] \equiv -\sum_y \sum_x f_{X,Y}(x,y) \log_2 f_{X|Y}(x,y)$$
$$= -\sum_y f_Y(y) \sum_x f_{X|Y}(x,y) \log_2 f_{X|Y}(x,y)$$

- Conditional entropy $H(X|Y)$ is average additional information, if Y is already known

$$\begin{aligned} H(X,Y) &= E\left[-\log_2 f_{X,Y}(X,Y)\right] \\ &= E\left[-\log_2 \left(f_Y(Y) f_{X|Y}(X,Y)\right)\right] \\ &= E\left[-\log_2 f_Y(Y)\right] + E\left[-\log_2 f_{X|Y}(X,Y)\right] \\ &= H(Y) + H(X|Y) \end{aligned}$$

Conditional entropy (cont.)

- Independent random variables X and Y

$$H(X|Y) = E\left[-\log_2 f_{X|Y}(X,Y)\right] = E\left[-\log_2 f_X(X)\right] = H(X)$$

- Let $\mathbf{Y}'$ be a random vector containing any subset of elements of another random vector $\mathbf{Y}$

$$H(X|\mathbf{Y}) \le H(X|\mathbf{Y}')$$

Equality, iff X is conditionally independent from elements missing from Y.

Hence, extra prior information can only reduce uncertainty.

Bit-rate bound for Markov random process

- Consider discrete, stationary Markov-p random process, characterized by conditional PMF $f_{X_p|\mathbf{X}_{0:(p-1)}}$

- Achievable lower bound for bit-rate: Shannon's Noiseless Source Coding Theorem:

$$R \geq H\left(X_p \mid \mathbf{X}_{0:(p-1)}\right)$$

- Important to exploit statistical dependence

$$H\left(X_p \mid \mathbf{X}_{0:(p-1)}\right) \leq H\left(X_p\right)$$

Switched variable length coding

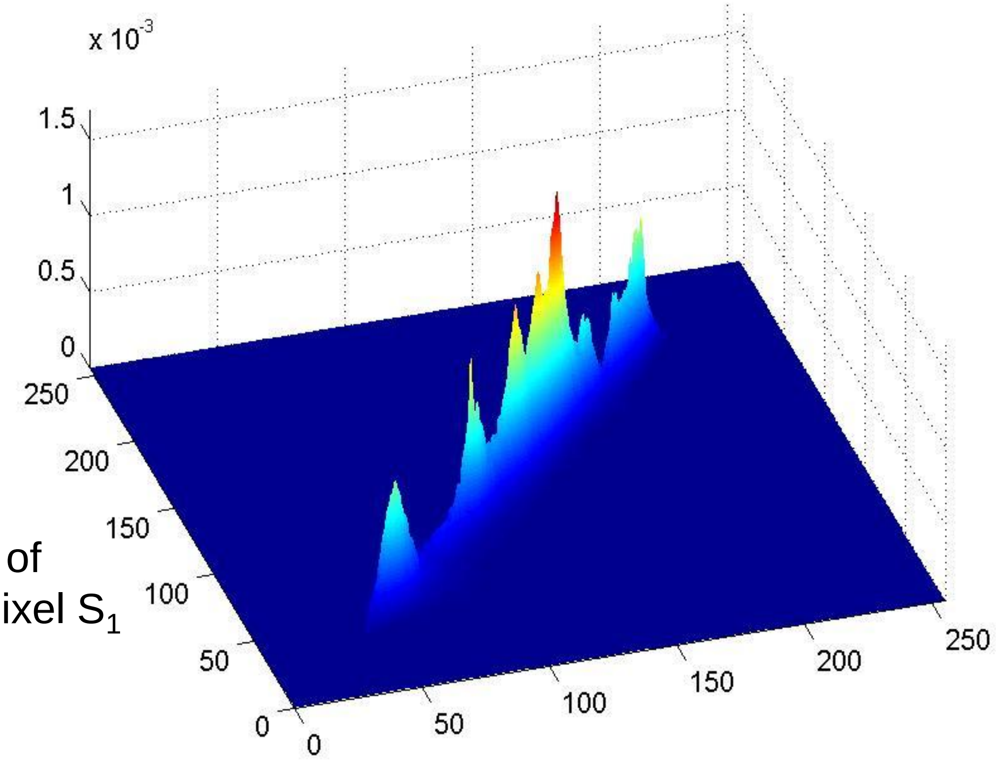
- How to encode $\{X_n\}$, such that the average code word length approaches $H\left(X_p \mid \mathbf{X}_{0:(p-1)}\right)$?
- Idea: switch among a set of variable length codes for X_n (e.g., Huffman or Golomb), one for each “state” $\mathbf{X}_{(n-p):(n-1)}$
- Number of states for 8-bit pixels

p = 1	256 states
p = 2	65,536 states
p = 3	16,777,216 states

Statistical Dependence of Adjacent Pixels

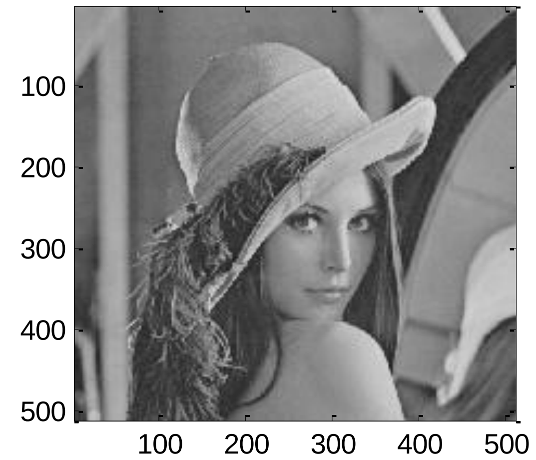
Histogram of two horizontally adjacent pixels
(‘Lena’, 512 x 512 pixels, 8 bpp)

PMF



Amplitude of adjacent pixel S_1

Amplitude of current pixel S_0



Calculating entropies for an individual image

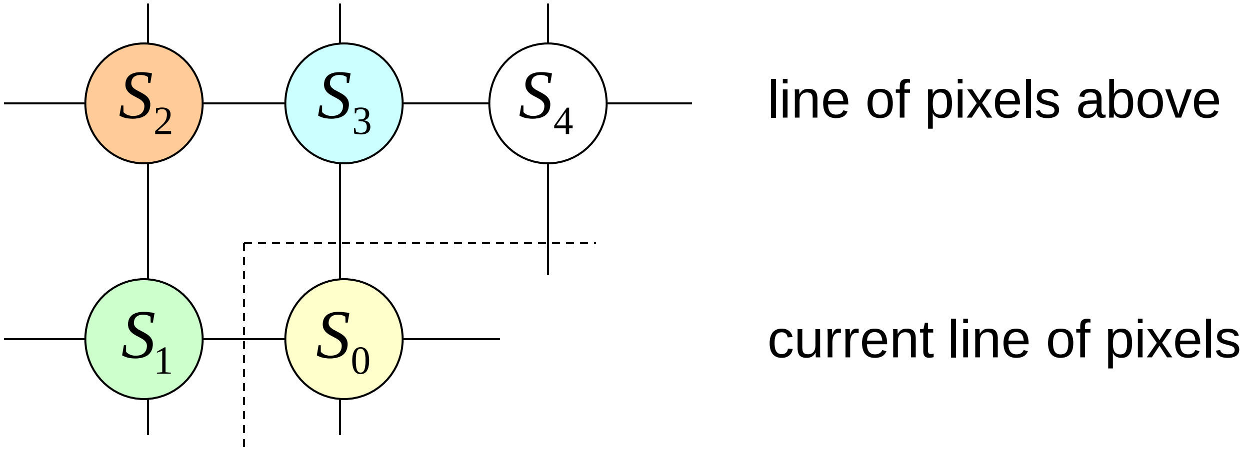
- Assume that a particular image has its own entropy code – implies some overhead to send code book
- Obtain probabilities from the image histogram
- Joint entropy for two adjacent pixels (bound for joint coding of 2x1 blocks)

$$H(S_0,S_1) = -\sum_{s_0}\sum_{s_1} f_{S_0,S_1}(s_0,s_1)\log_2 f_{S_0,S_1}(s_0,s_1)$$

- First-order conditional entropy (bound for conditional coding using previous pixel)

$$H(S_0 \mid S_1) = -\sum_{s_0}\sum_{s_1} f_{S_0,S_1}(s_0,s_1)\log_2 f_{S_0|S_1}(s_0 \mid s_1)$$

Conditional entropy for an image



component	$H(S_0)$	$H(S_0 \mid S_1)$	$H(S_0 \mid S_3)$	$H(S_0 \mid S_2)$
Y	7.23	4.67	4.32	4.86
Cb	5.47	3.80	3.58	3.85
Cr	5.42	3.69	3.55	3.82

Image: ‘Lena’, 512 x 512 pixels, 8 bpp
All values in bpp

Cross Entropy

- Assume that an entropy code is used which is optimal for probabilities $g_{S_0,S_1}(s_0,s_1)$ rather than the "true" probabilities
- Obtain "true" image probabilities $f_{S_0,S_1}(s_0,s_1) \neq g_{S_0,S_1}(s_0,s_1)$ from histogram
- Joint cross entropy for two adjacent pixels (bound for coding of 2x1 blocks)

$$H_{cross}(f_{S_0,S_1} \| g_{S_0,S_1}) = E_{f_{S_0,S_1}}\left\{-\log_2 g_{S_0,S_1}(s_0,s_1)\right\} - \sum_{s_0}\sum_{s_1} f_{S_0,S_1}(s_0,s_1)\log_2 g_{S_0,S_1}(s_0,s_1)$$

- First-order conditional cross entropy (bound for coding using previous pixel)

$$H_{cross}(f_{S_0|S_1} \| g_{S_0|S_1}) = -\sum_{s_0}\sum_{s_1} f_{S_0,S_1}(s_0,s_1)\log_2 g_{S_0|S_1}(s_0,s_1)$$

Kullback Leibler divergence

- Kullback-Leibler divergence (increase due to not using "true" probabilities)

$$D_{KL}(f \| g) = H_{cross}(f \| g) - H_{cross}(f \| f) \geq 0$$

- Not symmetric

$$D_{KL}(f \| g) \neq D_{KL}(g \| f)$$

- Also known as "relative entropy"

Run-length coding

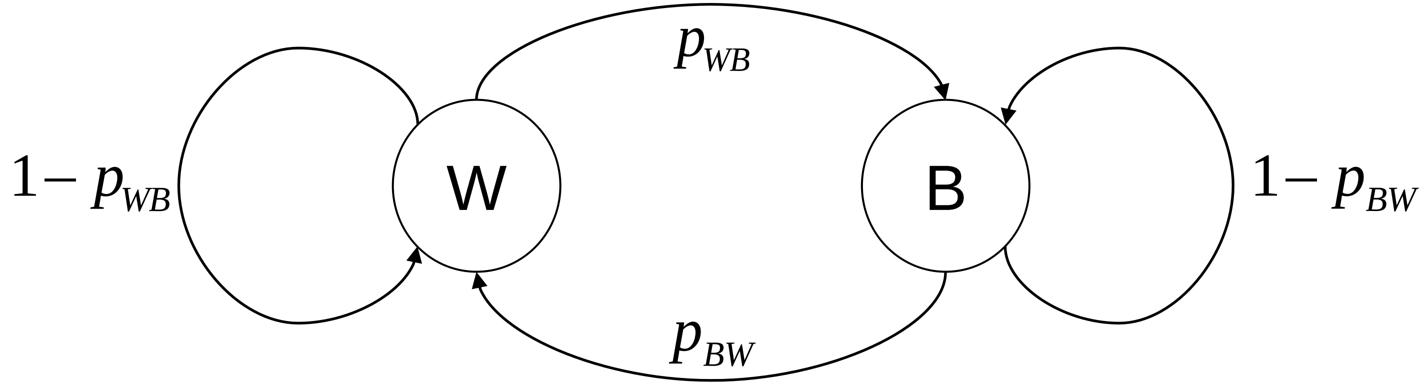
- For sources that emit “runs” of identical symbols
- Replace a sequence $\{x_n\}$ by a shorter sequence of symbol pairs $\{a_k, r_k\}$ such that

$$x_n = a_k \quad \text{for all } n : \sum_{j=1}^{k-1} r_j < n \le \sum_{j=1}^{k} r_j$$

- Entropy coding (e.g., Huffman coding) of new symbol pairs $\{a_k, r_k\}$
- For binary images, a_k can be omitted.

Statistical model for binary images of line-art

- Markov-1 model *[Capon, 1959]*



- State probabilities

$$\Pr\{W\} = 1 - \Pr\{B\} = \frac{p_{BW}}{p_{BW} + p_{WB}}$$

- Run-length distributions

$$\Pr\{k \text{ successive white pixels}\} = (1 - p_{WB})^{k-1}\, p_{WB} \quad \text{for } k = 1, 2, 3, \ldots$$

$$\Pr\{k \text{ successive black pixels}\} = (1 - p_{BW})^{k-1}\, p_{BW} \quad \text{for } k = 1, 2, 3, \ldots$$

- Golomb code for geometric distribution

Measured parameters for Capon model

Document	Weather Map	Printed Text
$\Pr\{W\}$	0.887	0.935
$\Pr\{B\}$	0.113	0.065
p_{WB}	0.027	0.024
p_{BW}	0.214	0.347
$H(X_i \mid X_{i-1})$	0.241 bpp	0.215 bpp

[Kunt,1974]

Facsimile compression standards

- Standards by the ITU-T (formerly CCITT)
  - T.4 (Group 3)
    - used by all fax machines over PSTN (public switched telephone network)
    - 1-d modified Huffman code (MH) or 2-d MMR code optional
  - T.6 (Group 4): fax over digital networks (e.g., ISDN)
    - Always 2-d MMR code
    - Less error-resilient than Group 3
- Picture formats
  - Horizontal resolution: 1728 pixels/line (1664 active) → 8.05 pixel/mm
  - Vertical resolution
    - Standard mode: 3.85 lines/mm (978 lines/page)
    - Fine mode: 7.7 lines/mm (1956 lines/page)
    - Very-fine mode: 15.4 lines/mm (3912 lines/pages)

Group 3 fax: modified Huffman code

- Lengths of runs of white pixels and black pixels encoded within a scan line
- Each run represented as

$$\text{white runs: } r_w = 64 \times r_{w/make-up} + r_{w/term} \qquad \text{black runs: } r_b = 64 \times r_{b/make-up} + r_{b/term}$$

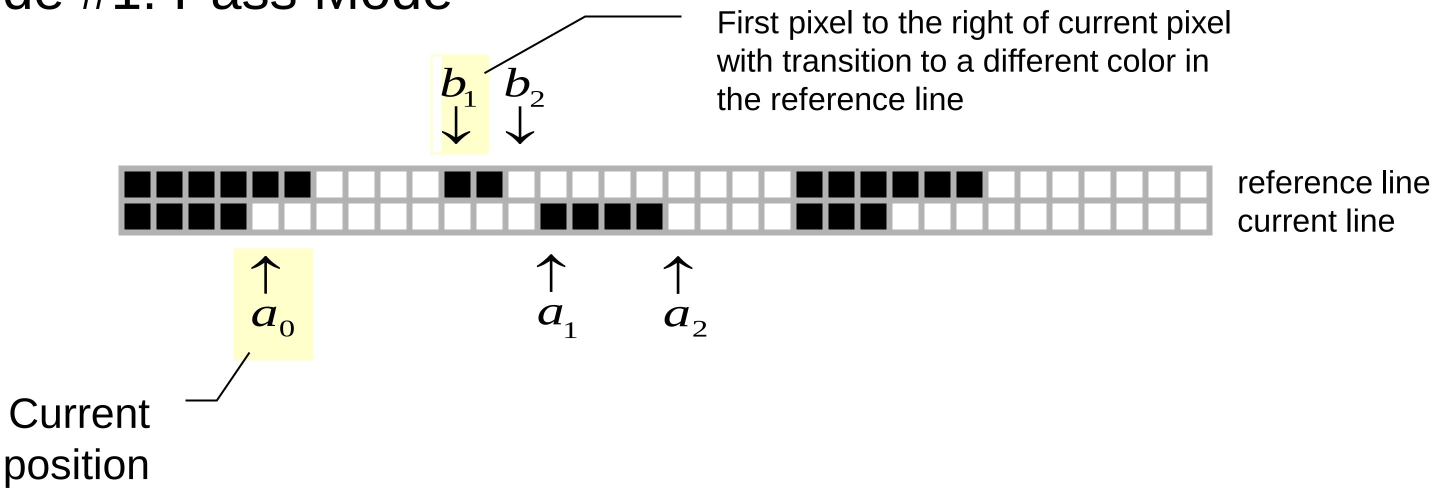
- "Make-up" and "termination" run-lengths encoded independently
- Two separate Huffman code tables for white and black runs ("make-up" and "termination" codes) based on the statistics of 8 representative documents
- Shortest code words (2 bits) for black runs of length 2 and 3
- Shortest code words (4 bits) for white runs of lengths 2 . . . 7
- Same code tables also used as part of Group 4 fax standard (MMR)
- Special EOL codeword for each line, 6x EOL as end of page

2-d fax coding

- MMR – Modified modified READ (relative address designate)
- Optional enhancement for Group 3, mandatory for Group 4
- For error resilience, Group 3 standard requires that at most K-1 lines are encoded 2-d, with $K=2$ for standard resolution and $K=4$ for fine resolution.

Modified modified READ algorithm

- Encode black and white run lengths relative to reference line above
- Mode #1: Pass Mode

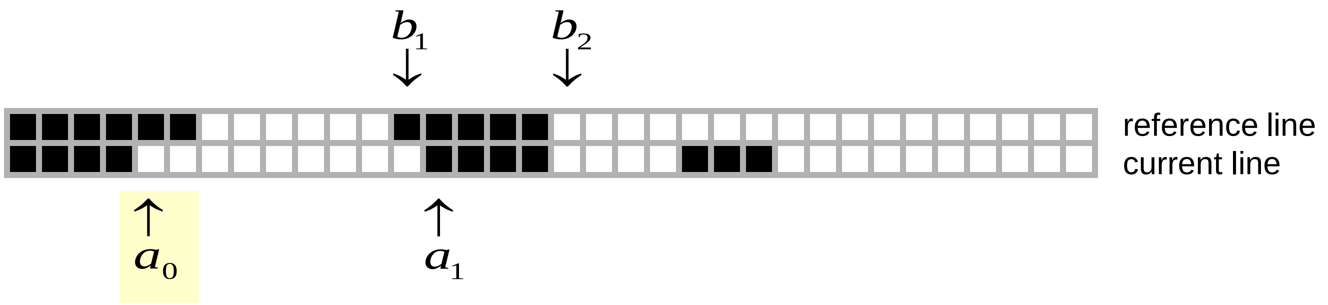


If $b_2 < a_1$: issue pass code word 0001

Update pointers: $a_0 \leftarrow b_2$, find new b_1, b_2

Modified modified READ algorithm (cont.)

- Mode #2: Vertical Mode



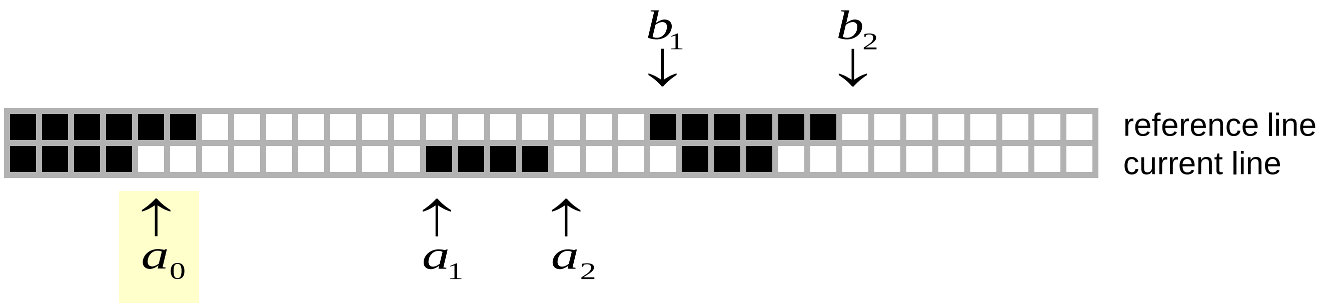
If $|a_1 - b_1| \leq 3$: encode difference $a_1 - b_1$ using table

$a_1 - b_1$	-3	-2	-1	0	1	2	3
	0000011	000011	011	1	010	000010	0000010

Update pointers: $a_0 \leftarrow a_1$, $b_1 \leftarrow b_2$ etc.

Modified modified READ algorithm (cont.)

- Mode #3: Horizontal Mode



If neither Pass Mode nor Vertical Mode:
encode next two runs 1-d, using Group 3 modified Huffman code

Update pointers: $a_0 \leftarrow a_2$, find new a_1, a_2, b_1, b_2

Compression efficiency of fax standards

CCITT Document Number	Standard resolution (2,052,864 pixels)		Fine resolution (4,105,728 pixels)	
	1-d MHC	2-d READ	1-d MHC	2-d READ
1	133,095	93,196	266,283	141,826
2	123,930	53,366	247,443	80,550
3	244,028	138,411	487,485	229,375
4	436,450	366,055	871,983	553,942
5	253,509	162,186	506,283	257,548
6	191,347	78,577	381,905	132,509
7	428,028	357,130	855,841	539,152
8	238,221	89,654	476,624	137,560
Average	256,076	167,322	511,731	259,058

[Yasuda, 1980]

Reading

- Wiegand + Schwarz, Chapter 3
- Taubman + Marcellin, Chapter 2.3 – 2.5
- H. Meyr, H. G. Rosdolsky, T.S. Huang, Optimum Run Length Codes, IEEE Trans. Communications, vol. COM-22, no. 6, pp. 1425-1433, June. 1974.
- H. G. Musmann, D. Preuss, Comparison of Redundancy Reducing Codes for Facsimile Transmission of Documents, IEEE Trans. Communications, vol. COM-25, no. 11, pp. 1425-1433, Nov. 1977.