Assignment

Read Section 5.1 page 125-138
Homework 1 due April 11

Today's Topics

Simple Description of Excitation
Block Echo Neglecting $T_1, T_2$
Small - TI - Angle Solution
Excitation E - space
Slice Selective Fourier Descom
Simple Description of Excitation

Polarized Equilibrium

Lab Ref Frame

Rotating Ref Frame

Motion of the magnetization described by the Bloch equation

Bloch Equation including Relaxation

\[
\frac{d}{dt} \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{T_2} & \delta G \cdot \nabla & -\delta B \cdot \nabla \\ -\delta G \cdot \nabla & -\frac{1}{T_2} & \gamma B \cdot \nabla \\ \gamma B \cdot \nabla & -\gamma B \cdot \nabla & -\frac{1}{T_1} \end{pmatrix} \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} + M_0 \begin{pmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{pmatrix}
\]

Where

\( \nabla = (G_x, G_y, G_z) \) Gradient (GCEs)

\( \vec{r} = (x, y, z) \) Position

For most of this course, we will solve \( T_1, T_2 \). Excitation is fast compared to relaxation.
**Bloch Equation Neglecting Relaxation**

\[ \frac{1}{T_1}, \frac{1}{T_2} \approx 0 \]

\[
\begin{pmatrix}
\frac{d}{dt}(M_x) \\
\frac{d}{dt}(M_y) \\
\frac{d}{dt}(M_z)
\end{pmatrix} =
\begin{pmatrix}
0 & \delta B_{1x} & -\delta B_{1y} \\
-\delta B_{1x} & 0 & \delta B_{1z} \\
\delta B_{1y} & -\delta B_{1z} & 0
\end{pmatrix}
\begin{pmatrix}
M_x \\
M_y \\
M_z
\end{pmatrix}
\]

The solution to this equation is a rotation.

We will return to this in 2 weeks!

**Small-TIP-Angle Approximation**

Flip angle & small means

\[ M_z \approx M_0 \]

First two equations decouple

\[
\frac{d}{dt} M_x = 0 + \delta B_{1x} M_y - \delta B_{1y} M_0 \\
\frac{d}{dt} M_y = -\delta B_{1x} M_x + 0 + \delta B_{1z} M_0
\]
Combine these into a single eqn

**Define**

\[ m_{xy} = m_x + i m_y \quad (m_{xy}(t, \xi)) \]
\[ B_1 = B_{1x} + i B_{1y} \quad (B_1(t)) \]

**Multiply the second \( \frac{d}{d \xi} m_y \) equation by \( i \), and add the two**

\[
\frac{d}{d \xi} (m_x + i m_y) = \delta \xi \cdot \nabla (m_y - i m_x) + \delta m_0 (-B_{1y} + i B_{1x})
\]
\[
= \delta \xi \cdot \nabla (-i^2 m_y - i m_x) + \delta m_0 (i^2 B_{1y} + i B_{1x})
\]
\[
= \delta \xi \cdot \nabla (-i)(m_x + i m_y) + \delta m_0 (i)(B_{1x} + i B_{1y})
\]

**Equation**

\[
\frac{d}{d \xi} m_{xy} = -i \delta \xi \cdot \nabla m_{xy} + i m_0 \delta B_1
\]

**Boundary Condition**

\[
\frac{d}{d \xi} m_{xy}(\xi, t) = -i \delta \xi \cdot \nabla m_{xy}(\xi, t) + i m_0 \delta B_1(t)
\]

**Initial Condition**

\[
\frac{d}{d \xi} m_{xy}(\xi, \xi_0) + i \delta \xi \cdot \nabla m_{xy}(\xi, \xi_0) = i m_0 \delta B_1(\xi_0)
\]

**Simple 1st Order Differential Eqn**

3
Solve using integrator factor

Multiply both sides by

\[ e^{i \int_0^t \gamma(s) \cdot \omega ds} \]

Producing

\[ m_{xy}(r,t) e^{i \int_0^t \gamma(s) \cdot \omega ds} + i \delta B_c(t) \cdot \varepsilon \quad m_{xy}(r,t) e^{i \int_0^t \gamma(s) \cdot \omega ds} = i \delta m_0 B_c(t) e^{i \int_0^t \gamma(s) \cdot \omega ds} \]

The left side is now an exact derivative

\[ \frac{d}{dt} \left[ m_{xy}(r,t) e^{i \int_0^t \gamma(s) \cdot \omega ds} \right] \]

\[ = i \delta m_0 B_c(t) e^{i \int_0^t \gamma(s) \cdot \omega ds} \]

Integrating both sides

\[ m_{xy}(r,t) e^{i \int_0^t \gamma(s) \cdot \omega ds} = i m_0 \int_0^t \delta B_c(t) e^{i \int_0^t \gamma(s) \cdot \omega ds} \, dt \]

\[ m_{xy}(r,t) = i m_0 e^{-i \int_0^t \gamma(s) \cdot \omega ds} \int_0^t \delta B_c(t) e^{i \int_0^t \gamma(s) \cdot \omega ds} \, dt \]

\[ = i m_0 \int_0^t \delta B_c(t) e^{-i \int_0^t \gamma(s) \cdot \omega ds} e^{i \int_0^t \gamma(s) \cdot \omega ds} \, dt \]
**Limits on the Arguments of Exponentials**

\[ \gamma_t \]

\[ (0, \rho) \quad (0, t) \]

\[ \varphi (\rho, t) \quad \varphi \]

INTEGRATION TIME WE ARE EVALUATING \( m_{xy} \)

**Combining Exponentials**

\[ m_{xy} (\rho, t) = i \mu_0 \int_{-\omega}^{t} \varphi (\rho, s) e^{-i \int_{\rho}^{t} \varphi (\rho, s) \omega ds} d\rho \]

**This is in the Form of a Fourier Transform**

**K-space Formulation**

**Define**

\[ k (\rho, t) = -\frac{i}{2\pi} \int_{-\omega}^{t} \varphi (\rho, s) d\rho \]

(INTEGRATE OF REMAINING GRADIENT)

**Different from Readout Convention for Excitation**

"Excitation" \( k \)-space

**Then**

\[ m_{xy} (\rho, t) = i \mu_0 \int_{-\omega}^{t} \varphi (\rho, s) e^{-i \int_{\rho}^{t} \varphi (\rho, s) \omega ds} d\rho \]
**K-space is in phase of gradient from end of excitation to readout sample.**

\[ K_r(t) = \frac{\sigma}{2\pi} \int_0^t G(\tau) ds \]

**Signal Equation**

\[ s(t) = \int R_k \cdot m_{xy}(r) e^{-2\pi i \xi(t)} \cdot d\xi \]

Compare to readout K-space
**Example: Slice Selection**

\[ B_1(t) \]

\[ G_z(t) \]

\[ K(t, t) = \frac{D}{\pi^2} \int_0^t G_z(s) ds \]

**RF Applies a Weighting in k-space During (1)**

**The Refocusing Lobe (2) Shifts the Weighting Back to the Middle**

**Refocuses the Slice**
SMALL INCREMENT IN EXCITATION
\( \delta B, (\tau) \Delta t \)

PRODUCES SMALL INCREMENT IN MAGNETIZATION
\( \Delta m_{xy} = (\delta B, (\tau) \Delta t) (i \omega_0) \)

THIS PROCEEDS BY
\[ k(\tau, t) = -\frac{\delta}{2\pi} \int_{-\infty}^{t} G_2(s) ds \]

TO PRODUCE A PHASE
\[ e^{i \omega c(t, \tau) t} \]

MAGNETIZATION FROM THIS INCREMENT AT ENDS OF PULSE
\[ (i \omega_0) \delta B, (\tau) \Delta t e^{i \omega c(t, \tau) t} \]
INTEGRATE OVER ALL SUCH SCALES:

\[ m_{xy}(z, t) = i m_0 \int_{-\infty}^{t} x_{3, \tau}(\tau) e^{i 2 \pi k (z - \tau)} \, d\tau \]
Fourier Design of Slice Selective Pulses

Slice Profile is Fourier Transform of RF Pulse, k-Space Weighing.

Choose an RF Pulse with MCZ Transform SINC

\[ P_c(\xi) = \text{sinc}(\xi) \]

Not Practical, since sinc(.) continues indefinitely

Truncated SINC

\[ \text{sinc}(\xi) \text{ rect}(\frac{\xi}{2N}) \]

Too much ripple

Windowed SINC

\[ \text{sinc}(\xi) \text{ Hamming Window} \]

Just right!
CHARACTERIZATION OF PULSE SHAPE

Time-Bandwidth Product

\[ T \times B = (2N)^1 \]
\[ = 2N \quad \text{total number of zeros} \]

Typical Pulses

- T\( B \times W = 2 \)
- T\( B \times W = 4 \)
- T\( B \times W = 8 \)
- T\( B \times W = 12 \)

SSFP a's 180° 90° α's 8° Sat Pulses Slab Select
msinc = \( \frac{1}{2} \) msinc = 1 msinc = 2 msinc = 3

If we fix bandwidth, and make T lower:

- T\( B \times W = 4 \)
- T\( B \times W = 8 \)

More selective profiles
If we fix duration, and increase bandwidth, the excitation becomes wider.

Typically in MRI, we fix duration, and adjust the gradient amplitude to compensate for the increased bandwidth.
**Example**

We want a TRW=8 (msinc 2) pulse with a 2 ms duration.

If the slice thickness is 1 cm, what is the gradient amplitude?

**Answer:**

\[ (T \times R_w) = 8 \]
\[ (2 \text{ms}) \times (R_w) = 8 \]

\[ R_w = 4 \text{ kHz} \]

We want this to correspond to a slice thickness \( \Delta z = 1 \text{ cm} \)

\[ \frac{6}{2\pi} G \Delta z = 4 \text{ kHz} \]

\[ \left( 4.257 \frac{\text{kHz}}{G} \right) G (1\text{cm}) = 4 \text{ kHz} \]

\[ G = 0.94 \text{ G/cm} \]