

Final Examination

This is a 180 minute in-class final examination.

You may use two pieces of paper (double-sided), but otherwise this examination is closed book. While you may ask us questions if you find a question confusing, we've tried pretty hard to make the exam unambiguous and clear, so we're unlikely to say much.

Please respect the honor code.

All problems have equal weight. Some are (quite) straightforward. Others, not so much.

Some problems involve applications. But you do not need to know *anything* about the problem area to solve the problem; the problem statement contains everything you need.

The problems do *not* appear in order of increasing difficulty.

Name: _____

Stanford ID #: _____

F.1 *Some invertibility properties.*

- (a) Which of the following are equivalent for a square matrix A ? (i) A has linearly independent columns, (ii) A has linearly independent rows, (iii) A has a left inverse, (iv) A has a right inverse.

- (b) For each of the following matrices, identify whether the matrix has a left inverse, a right inverse, both left and right inverses, neither, or whether you cannot determine from the information provided.

- (i) A matrix A such that A is square and has linearly independent columns.

- (ii) The matrix B defined by

$$B = \begin{bmatrix} A \\ C \end{bmatrix}$$

where A is square and has independent rows.

- (iii) The matrix B defined by

$$B = \begin{bmatrix} A & C \end{bmatrix}$$

where A is square and has independent rows.

- (iv) The matrix B defined by

$$B = \begin{bmatrix} A & C^T \\ C & D \end{bmatrix}$$

where A and D are symmetric.

F.2 An $n \times n$ symmetric matrix P is a *projection matrix* if $P^2 = P$, that is, $PP = P$.

(a) Let u be a vector with $\|u\| = 1$. Show that $P = uu^T$ is a projection matrix.

(b) Let U be an $n \times k$ matrix with orthonormal columns. Is $P = UU^T$ a projection matrix? Why or why not?

(c) Give two non-zero projection matrices P_0 and P_1 such that $P_0P_1 = \mathbf{0}$.

F.3 Let u and v be 2-vectors with $\|u\| = \|v\| = 1$, where $u^T v > 0$ but $u \neq v$. Consider the following *alternating projection* scheme: beginning from a non-zero 2-vector $x_0 \in \mathbf{R}^2$, for $k = 0, 1, 2, 3, \dots$, we iterate

$$x_{k+1} = \begin{cases} uu^T x_k & \text{if } k \text{ is even} \\ vv^T x_k & \text{if } k \text{ is odd,} \end{cases}$$

which generates a sequence of 2-vectors, beginning with $x_1 = uu^T x_0$. Draw a picture of this iteration. Your picture should include u, v , the lines passing through u and v , and several points $x_0, x_1, x_2, \dots$ in the iteration.

F.4 *Block matrix inversion..* Let $A \in \mathbf{R}^{n \times n}$, $U \in \mathbf{R}^{n \times k}$, and $V \in \mathbf{R}^{k \times n}$, where A is invertible. Assume that $I + VA^{-1}U$ is invertible. Show the *Sherman-Morrison-Woodbury Formula* that

$$(A + UV)^{-1} = A^{-1} - A^{-1}U(I + VA^{-1}U)^{-1}VA^{-1}. \quad (1)$$

Hint. Consider the block matrix formula

$$\begin{bmatrix} A & U \\ V & -I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Write this as two equations involving A, U, V, X, Y , and identity matrices I . Solve for X in terms of Y in the first row of the block formula above, and then solve for Y in the second row. Substitute this back to get X . Why is $X = (A + UV)^{-1}$?

F.5 Consider a (two-dimensional system of a) ball launched into the air (ignoring air resistance) with initial velocity vector $v \in \mathbf{R}^2$ denoting its horizontal and vertical speeds. The position $p(t)$ of the ball at time t is then

$$p(t) = tv - \frac{a}{2} \begin{bmatrix} 0 \\ t^2 \end{bmatrix},$$

where $a > 0$ is the acceleration. We (noisily) measure 2-dimensional positions $p_1, p_2, \dots, p_k \in \mathbf{R}^2$ of the ball at (positive scalar) times $t_1, t_2, \dots, t_k$ seconds after the initial launch.

- (a) Assume the initial position $p_0 = \mathbf{0}$ is known. We estimate the initial velocity v of the ball by choosing the $\hat{v}$ minimizing

$$\left\| t_1 v - \frac{a}{2} \begin{bmatrix} 0 \\ t_1^2 \end{bmatrix} - p_1 \right\|^2 + \left\| t_2 v - \frac{a}{2} \begin{bmatrix} 0 \\ t_2^2 \end{bmatrix} - p_2 \right\|^2 + \dots + \left\| t_k v - \frac{a}{2} \begin{bmatrix} 0 \\ t_k^2 \end{bmatrix} - p_k \right\|^2$$

in the variable v . Formulate this as a least squares problem of minimizing

$$\|Av - b\|^2.$$

Give your matrix A and vector b , and specify their dimensions. How many measurements k are necessary for this problem to have a unique solution? It may be useful to recall that for vectors x, y, z, w of appropriate sizes,

$$\|x - y\|^2 + \|z - w\|^2 = \left\| \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} y \\ w \end{bmatrix} \right\|^2.$$

- (b) We now assume the initial position $p \in \mathbf{R}^2$ is unknown and wish to recover it along with the initial velocity. Noting that $p_i = p + vt_i - (0, \frac{at_i^2}{2})$, we can do this by finding the $\hat{p} \in \mathbf{R}^2$ and $\hat{v} \in \mathbf{R}^2$ minimizing

$$\left\| p + t_1 v - \frac{a}{2} \begin{bmatrix} 0 \\ t_1^2 \end{bmatrix} - p_1 \right\|^2 + \left\| p + t_2 v - \frac{a}{2} \begin{bmatrix} 0 \\ t_2^2 \end{bmatrix} - p_2 \right\|^2 + \cdots + \left\| p + t_k v - \frac{a}{2} \begin{bmatrix} 0 \\ t_k^2 \end{bmatrix} - p_k \right\|^2$$

in the variables p and v . Formulate this as a least squares problem of minimizing

$$\left\| A \begin{bmatrix} p \\ v \end{bmatrix} - b \right\|^2.$$

Give your matrix A and vector b , and specify their dimensions. How many measurements k are necessary for this to have a unique solution?

F.6 The law of gravity is that the gravitational force F between (point) masses M_0 and M_1 at a distance r from one another is

$$F = G \frac{M_0 M_1}{r^2},$$

where $G \approx 6.674 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$ is the gravitational constant. The force on a body of mass M in constant rotation at a velocity (really, tangential speed) v required to keep it in circular motion at distance r from a center point is

$$F_{\text{rad}} = \frac{Mv^2}{r}.$$

We believe *dark matter* exists because the force of gravity is not enough to keep stars in rotation around galactic centers—they should fly off into space. (They don't.)

Assume we have m measurements $M_1, M_2, \dots, M_m$ of masses and associated tangential speeds $v_1, v_2, \dots, v_m$, along with their distances $r_1, \dots, r_m$ from a galactic center with observable mass M_0 . We wish to find the hidden mass H that would be required at the galactic center to keep the stars in orbit, that is, to find the mass H so that the equalities

$$G \frac{(M_0 + H)M_i}{r_i^2} = \frac{M_i v_i^2}{r_i}$$

are all (approximately) satisfied. Give a least squares formulation for finding an $\hat{H}$ (approximately) satisfying these equalities, and give an explicit solution for $\hat{H}$ from your formulation.

F.7 Finding the dark matter. Instead of assuming dark matter is concentrated in a point mass at the center of the galaxy, we instead wish to find the locations of this dark matter. As in Question F.6 we observe m masses $M_1, \dots, M_m$ in circular motion at speeds $v_1, \dots, v_m$ and distances $r_1, \dots, r_m$ from a central mass M_0 . (See Figure 1.) Unknown hidden masses $h_1, \dots, h_k$ are distributed through space (you may think of these as being in a grid; we wish to estimate the amount of dark matter in each grid square), where hidden mass h_j is at (known) distance r_{ij} from mass M_i , and the line between the location of h_j and mass M_i forms known angle θ_{ij} from the radial. Mass h_j thus exerts tangential and radial forces

$$F_{ij,\text{tan}} = G \frac{h_j M_i}{r_{ij}^2} \sin \theta_{ij} \quad \text{and} \quad F_{ij,\text{rad}} = G \frac{h_j M_i}{r_{ij}^2} \cos \theta_{ij},$$

respectively, on mass M_i . To remain in constant motion, the tangential forces must sum to 0, while the centripetal forces $F_{ij,\text{rad}}$ must balance the speed of mass M_i , so that we have

$$\sum_{j=1}^k F_{ij,\text{tan}} = 0 \quad \text{and} \quad \sum_{j=1}^k F_{ij,\text{rad}} + G \frac{M_0 M_i}{r_i^2} = \frac{M_0 v_i^2}{r_i} \quad (2)$$

for each $i = 1, \dots, m$. As measurements are noisy and approximate, we rarely have exact equality above. Formulate finding the vector $h = (h_1, \dots, h_k)$ of masses that (approximately) solve the equations of motion (2) as a least-squares problem in the form

$$\text{minimize } \|Ah - b\|^2.$$

Specify your matrix A , the vector b , and their sizes. (Write your answer on the next page.)

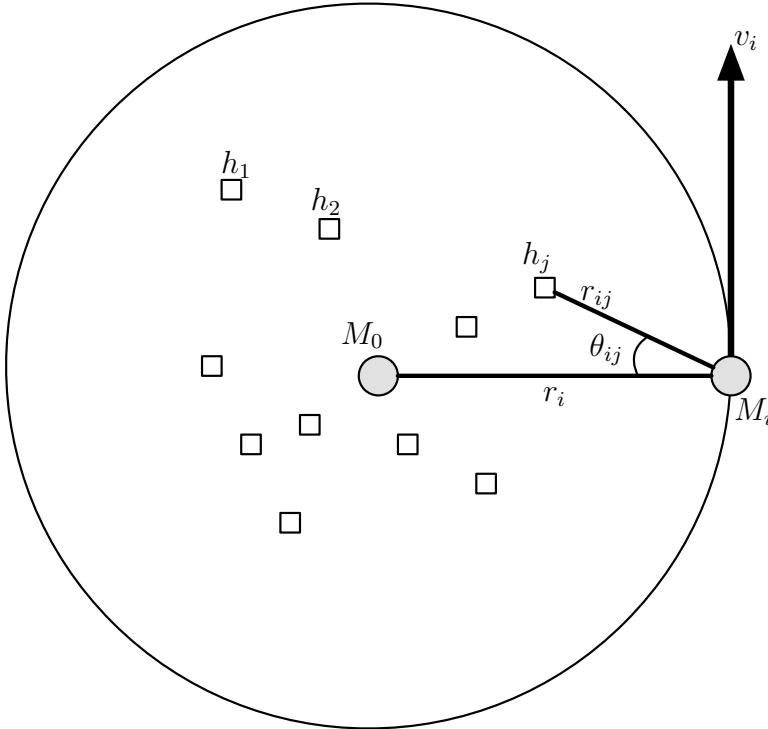


Figure 1: Motion diagram for mass M_i with mass M_0 at center (small grey circles). Hidden masses $h_1, \dots, h_k$ are represented as squares. The mass M_i is at distance r_i from the center, traveling at tangential speed v_i . Its angle to the position of hidden mass h_j is θ_{ij} , and the distance of mass M_i to hidden mass h_j is r_{ij} .

Answer Question F.7 here.

F.8 Extra credit. *Updating a least squares solution with a single example.* Let $\hat{x} = (A^T A)^{-1} A^T b$ be the least squares solution to the problem of minimizing $\|Ax - b\|^2$, and for a vector $a \in \mathbf{R}^n$ and scalar $b_{m+1} \in \mathbf{R}$ define

$$C = \begin{bmatrix} A \\ a^T \end{bmatrix} \in \mathbf{R}^{m+1 \times n} \quad \text{and} \quad d = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix}.$$

Justify the following *rank one update* for $\hat{y} = (C^T C)^{-1} C^T d$: that

$$\hat{y} = \hat{x} + \frac{(A^T A)^{-1} a}{1 + a^T (A^T A)^{-1} a} (b_{m+1} - a^T \hat{x}).$$

If you had already computed $(A^T A)^{-1}$, roughly how many flops would computing $\hat{y}$ require?
Hint. Use the Sherman-Morrison-Woodbury formula (1) in Question F.4.