ORBITAL MECHANICS

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http://www.braeunig.us/space/orbmech.htm

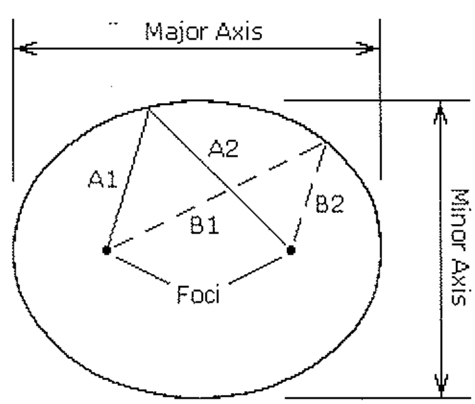
Orbital mechanics, also called flight mechanics, is the study of the motions of artificial satellites and space vehicles moving under the influence of forces such as gravity, atmospheric drag, thrust, etc. Orbital mechanics is a modern offshoot of celestial mechanics which is the study of the motions of natural celestial bodies such as the moon and planets. The root of orbital mechanics can be traced back to the 17th century when mathematician Isaac Newton (1642-1727) put forward his laws of motion and formulated his law of universal gravitation. The engineering applications of orbital mechanics include ascent trajectories, reentry and landing, rendezvous computations, and lunar and interplanetary trajectories.

Orbital Elements

To mathematically describe an orbit one must define six quantities, called *orbital elements*. They are

- Semi-Major Axis
- Eccentricity
- Inclination
- Argument of Periapsis
- Time of Periapsis Passage
- Longitude of Ascending Node

An orbiting satellite follows an oval shaped path known as an ellipse with the body being orbited, called the primary, located at one of two points called foci. An ellipse is defined to be a curve with the following property: for each point on an ellipse, the sum of its distances from two fixed points, called foci, is constant (see figure to right). The longest and shortest lines that can be drawn through the center of an ellipse are called the major axis and minor axis, respectively. The *semi-major axis* is one-half of the major axis and represents a satellite's mean distance from its primary. *Eccentricity* is the distance between the foci divided by the length of the major axis and is a number between zero and one. An eccentricity of zero indicates a circle.

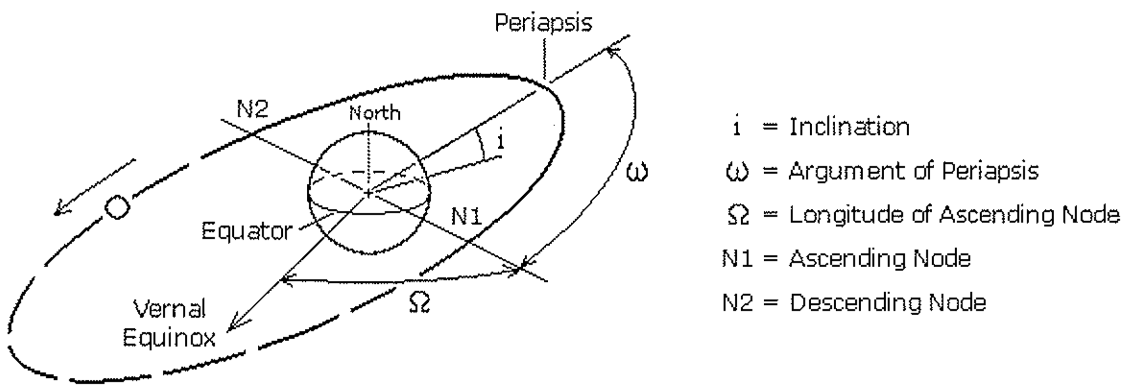


A1 + A2 = B1 + B2

Inclination is the angular distance between a satellite's orbital plane and the equator of its primary (or the ecliptic plane in the case of heliocentric, or sun centered, orbits). An inclination of zero degrees indicates an orbit about the primary's equator in the same direction as the primary's rotation, a direction called *prograde* (or direct). An inclination of 90 degrees indicates a polar orbit. An inclination of 180 degrees indicates a retrograde equatorial orbit. A *retrograde* orbit is one in which a satellite moves in a direction opposite to the rotation of its primary.

Periapsis is the point in an orbit closest to the primary. The opposite of periapsis, the farthest point in an orbit, is called *apoapsis*. Periapsis and apoapsis are usually modified to apply to the body being orbited, such as perihelion and aphelion for the Sun, perigee and apogee for Earth, perijove and apojove for Jupiter, perilune and apolune for the Moon, etc. The *argument of periapsis* is the angular distance between the ascending node and the point of periapsis (see figure below). The *time of periapsis passage* is the time in which a satellite moves through its point of periapsis.

Nodes are the points where an orbit crosses a plane, such as a satellite crossing the Earth's equatorial plane. If the satellite crosses the plane going from south to north, the node is the *ascending node*; if moving from north to south, it is the *descending node*. The *longitude of the ascending node* is the node's celestial longitude. Celestial longitude is analogous to longitude on Earth and is measured in degrees counter-clockwise from zero with zero longitude being in the direction of the vernal equinox.



In general, three observations of an object in orbit are required to calculate the six orbital elements. Two other quantities often used to describe orbits are period and true anomaly. *Period* is the length of time required for a satellite to complete one orbit. *True anomaly* is the angular distance of a point in an orbit past the point of periapsis, measured in degrees.

Types Of Orbits

For a spacecraft to achieve earth orbit, it must be launched to an elevation above the Earth's atmosphere and accelerated to orbital velocity. The most energy efficient orbit, that is one that requires the least amount of propellant, is a direct low inclination orbit. To achieve such an orbit, a spacecraft is launched in an eastward direction from a site near the Earth's equator. The advantage being that the rotational speed of the Earth contributes to the spacecraft's final orbital speed. At the United States' launch site in Cape Canaveral (28.5 degrees north latitude) a due east launch results in a "free ride" of 915 mph (1,470 kph). Launching a spacecraft in a direction other than east, or from a site far from the equator, results in an orbit of higher inclination. High inclination orbits are less able to take advantage of the initial speed provided by the Earth's rotation, thus the launch vehicle must provide a greater part, or all, of the energy required to attain orbital velocity. Although high inclination orbits are less energy efficient, they do have advantages over equatorial orbits for certain applications. Below we describe several types of orbits and the advantages of each:

Geosynchronous orbits (GEO) are circular orbits around the Earth having a period of 24 hours. A geosynchronous orbit with an inclination of zero degrees is called a *geostationary orbit*. A spacecraft in a geostationary orbit appears to hang motionless above one position on the Earth's equator. For this reason, they are ideal for some types of communication and meteorological satellites. A spacecraft in an inclined geosynchronous orbit will appear to follow a regular figure-8 pattern in the sky once every orbit. To attain geosynchronous orbit, a spacecraft is first launched into an elliptical orbit with an apogee of 22,240 miles (35,790 km) called a *geosynchronous transfer orbit* (GTO). The orbit is then circularized by firing the spacecraft's engine at apogee.

Polar orbits (PO) are orbits with an inclination of 90 degrees. Polar orbits are useful for satellites that carry out mapping and/or surveillance operations because as the planet rotates the spacecraft has access to virtually every point on the planet's surface.

Walking orbits: An orbiting satellite is subjected to a great many gravitational influences. First, planets are not perfectly spherical and they have slightly uneven mass distribution. These fluctuations have an effect on a spacecraft's trajectory. Also, the sun, moon, and planets contribute a gravitational influence on an orbiting satellite. With proper planning it is possible to design an orbit which takes advantage of these influences to induce a precession in the satellite's orbital plane. The resulting orbit is called a *walking orbit*, or precessing orbit.

Sun synchronous orbits (SSO) are walking orbits whose orbital plane precesses with the same period as the planet's solar orbit period. In such an orbit, a satellite crosses periapsis at about the same local time every orbit. This is useful if a satellite is carrying instruments which depend on a certain angle of solar illumination on the planet's surface. In order to maintain an exact synchronous timing, it may be necessary to conduct occasional propulsive maneuvers to adjust the orbit.

Hohmann transfer orbits are interplanetary trajectories whose advantage is that they consume the least possible amount of propellant. A Hohmann transfer orbit to an outer planet, such as Mars, is achieved by launching a spacecraft and accelerating it in the direction of Earth's revolution around the sun until it breaks free of the Earth's gravity and reaches a velocity which places it in a sun orbit with an aphelion equal to the orbit of the outer planet. Upon reaching its destination, the spacecraft must decelerate so that the planet's gravity can capture it into a planetary orbit.

To send a spacecraft to an inner planet, such as Venus, the spacecraft is launched and accelerated in the direction opposite of Earth's revolution around the sun (i.e. decelerated) until in achieves a sun orbit with a perihelion equal to the orbit of the inner planet. It should be noted that the spacecraft continues to move in the same direction as Earth, only more slowly.

To reach a planet requires that the spacecraft be inserted into an interplanetary trajectory at the correct time so that the spacecraft arrives at the planet's orbit when the planet will be at the point where the spacecraft will intercept it. This task is comparable to a quarterback "leading" his receiver so that the football and receiver arrive at the same point at the same time. The interval of time in which a spacecraft must be launched in order to complete its mission is called a *launch window.*

Newton's Laws of Motion and Universal Gravitation

Newton's laws of motion describe the relationship between the motion of a particle and the forces acting on it.

The first law states that if no forces are acting, a body at rest will remain at rest, and a body in motion will remain in motion in a straight line. Thus, if no forces are acting, the velocity (both magnitude and direction) will remain constant.

The second law tells us that if a force is applied there will be a change in velocity, i.e. an acceleration, proportional to the magnitude of the force and in the direction in which the force is applied. This law may be summarized by the equation

(1.1) $$F = ma$$

where F is the force, m is the mass of the particle, and a is the acceleration.

The third law states that if body 1 exerts a force on body 2, then body 2 will exert a force of equal strength, but opposite in direction, on body 1. This law is commonly stated, "for every action there is an equal and opposite reaction".

In his *law of universal gravitation*, Newton states that two particles having masses *m1* and *m2* and separated by a distance *r* are attracted to each other with equal and opposite forces directed along the line joining the particles. The common magnitude *F* of the two forces is

$$(1.2) \qquad F = G\left(\frac{m_1 m_2}{r^2}\right)$$

where *G* is an universal constant, called the *constant of gravitation*, and has the value 3.439×10^{-8} lb-ft^2/slug^2 in U.S. units and 6.673×10^{-11} N-m^2/kg^2 in SI units.

Let's now look at the force that the earth exerts on an object. If the object has a mass *m*, and the earth has mass *M*, and the object's distance from the center of the earth is *r*, then the force that the earth exerts on the object is GmM/r^2 . If we drop the object, the earth's gravity will cause it to accelerate toward the center of the earth. By Newton's second law (F = ma), this acceleration *g* must equal $(GmM/r^2)/m$, or

$$(1.3) \qquad g = \frac{GM}{r^2}$$

At the surface of the earth this acceleration has the valve 32.17 ft/s^2 in U.S. units and 9.807 m/s^2 in SI units.

Many of the upcoming computations will be somewhat simplified if we note that the product *GM* may be expressed as

$$GM = gR^2 = 32.17\,(3959 \times 5280)^2 = 1.408\times10^{16}\ \text{ft}^3/\text{s}^2, \quad \text{or}$$

$$GM = 9.807\,(6371 \times 1000)^2 = 3.986\times10^{14}\ \text{m}^3/\text{s}^2$$

where *R* is the radius of the earth, i.e. 3,959 miles or 6,371 kilometers.

Uniform Circular Motion

In the simple case of free fall, a particle accelerates toward the center of the earth while moving in a straight line. The velocity of the particle changes in magnitude, but not in direction. In the case of uniform circular motion a particle moves in a circle with constant speed. The velocity of the particle changes continuously in direction, but not in magnitude. From Newton's laws we see that since the direction of the velocity is changing, there is an acceleration. This acceleration, called *centripetal acceleration* is directed inward toward the center of the circle and is given by

$$(1.4) \qquad a = \frac{v^2}{r}$$

where *v* is the speed of the particle and *r* is the radius of the circle. Every accelerating particle must have a force acting on it, defined by Newton's second law (F = ma). Thus, a particle undergoing uniform circular motion is under the influence of a force, called *centripetal force*, whose magnitude is given by

$$(1.5) \qquad F = \frac{mv^2}{r}$$

The direction of *F* at any instant must be in the direction of *a* at the same instant, that is radially inward.

A satellite in orbit is acted on only by the forces of gravity. The inward acceleration which causes the satellite to move in a circular orbit is the gravitational acceleration caused by the body around which the satellite orbits. Hence, the satellite's centripetal acceleration is *g*, that is $g = v^2/r$. From Newton's law of universal gravitation we know that $g = GM/r^2$. Therefore, by setting these equations equal to one another we find that, for a circular orbit,

$$\frac{v^2}{r} = \frac{GM}{r^2}, \quad \text{or}$$

(1.6) $$v = \sqrt{\frac{GM}{r}}$$

Example 1.1: Calculate the velocity of an artificial satellite orbiting the earth in a circular orbit at an altitude of 120 miles above the earth's surface.

Given: r = (3,959 + 120) x 5,280 = 21,537,000 ft

From equation (1.6),

v = SQRT[GM / r]
v = SQRT[1.408x10^16 / 21,537,000]
v = 25,570 ft/s

Motions of Planets and Satellites

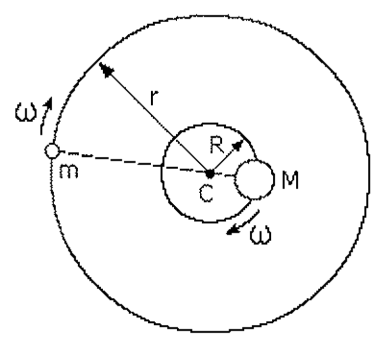
Through a lifelong study of the motions of bodies in the solar system, Johannes Kepler (1571-1630) was able to derive three basic laws known as *Kepler's laws of planetary motion* . Using the data compiled by his mentor Tycho Brahe (1546-1601), Kepler found the following regularities after years of laborious calculations:

1. All planets move in elliptical orbits with the sun at one focus.
2. A line joining any planet to the sun sweeps out equal areas in equal times.
3. The square of the period of any planet about the sun is proportional to the cube of the planet's mean distance from the sun.

These laws can be deduced from Newton's laws of motion and law of universal gravitation. Indeed, Newton used Kepler's work as basic information in the formulation of his gravitational theory.

As Kepler pointed out, all planets move in elliptical orbits, however, we can learn much about planetary motion by considering the special case of circular orbits. We shall neglect the forces between planets, considering only a planet's interaction with the sun. These considerations apply equally well to the motion of a satellite about a planet.

Let's examine the case of two bodies of masses M and m moving in circular orbits under the influence of each other's gravitational attraction. The center of mass of this system of two bodies lies along the line joining them at a point C such that $mr = MR$. The large body of mass M moves in an orbit of constant radius R and the small body of mass m in an orbit of constant radius r, both having the same angular velocity ω. For this to happen, the gravitational force acting on each body must provide the necessary centripetal acceleration. Since these gravitational forces are a simple action-reaction pair, the centripetal forces must be equal but opposite in direction. That is, $m\omega^2 r$ must equal $M\omega^2 R$. The specific requirement, then, is that the gravitational force acting on either body must equal the centripetal force needed to keep it moving in its circular orbit, that is



(1.7) $$\frac{GMm}{(R+r)^2} = m\omega^2 r$$

If one body has a much greater mass than the other, as is the case of the sun and a planet or the earth and a satellite, its distance from the center of mass is much smaller than that of the other body. If we assume that *m* is negligible compared to *M*, then *R* is negligible compared to *r*. Thus, equation (1.7) then becomes

$$GM = \omega^2 r^3 \qquad (1.8)$$

If we express the angular velocity in terms of the period of revolution, $\omega = 2\pi/P$, we obtain

$$GM = \frac{4\pi^2 r^3}{P^2}, \quad \text{or}$$

$$P^2 = \frac{4\pi^2 r^3}{GM} \qquad (1.9)$$

where *P* is the period of revolution. This is a basic equation of planetary and satellite motion. It also holds for elliptical orbits if we define *r* to be the semi-major axis of the orbit.

A significant consequence of this equation is that it predicts Kepler's third law of planetary motion, that is $P^2 \sim r^3$.

Example 1.2: Calculate the period of revolution for the satellite in problem 1.1.

Given: r = 21,537,000 ft

From equation (1.9),

P^2 = 4 x pi^2 x r^3 / GM

P = SQRT[4 x pi^2 x r^3 / GM]
P = SQRT[4 x pi^2 x 21,537,000^3 / 1.408x10^16]
P = 5,292 s

Example 1.3: Calculate the radius of orbit (in SI units) for a earth satellite in a geosynchronous orbit, where the earth's rotational period is 86,164 seconds.

Given: P = 86,164 s

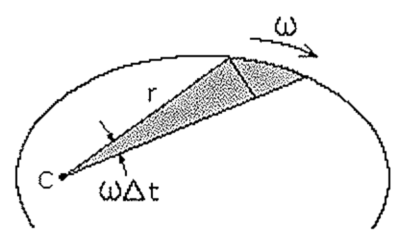
From equation (1.9),

P^2 = 4 x pi^2 x r^3 / GM

r = [P^2 x GM / (4 x pi^2)]^1/3
r = [86,164^2 x 3.986x10^14 / (4 x pi^2)]^1/3
r = 42,164,000 m

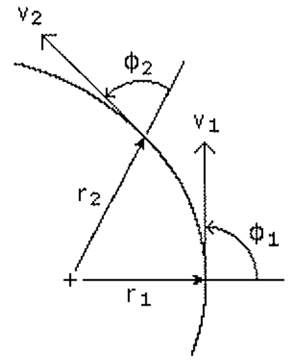
Kepler's second law of planetary motion must, of course, hold true for circular orbits. In such orbits both ω and r are constant so that equal areas are swept out in equal times by the line joining a planet and the sun. For elliptical orbits, however, both ω and r will vary with time. Let's now consider this case.

$$\lim_{t \to 0} \left[\frac{r(r\omega\Delta t)}{2}\right] = \frac{\omega r^2}{2} \qquad (1.10)$$



For any given body moving under the influence of a central force, the value ωr^2 is constant.

Let's now consider two points *P1* and *P2* in an orbit with radii *r1* and *r2*, and velocities *v1* and *v2*. Since the velocity is always tangent to the path, it can be seen that if ϕ is the angle between *r* and *v*, then



(1.11) $$v \sin\phi = \omega r$$

where *vsin*ϕ is the transverse component of *v*. Multiplying through by *r*, we have

(1.12) $$r v \sin\phi = \omega r^2 = \text{Constant}$$

or, for two points *P1* and *P2* on the orbital path

(1.13) $$r_1 v_1 \sin\phi_1 = r_2 v_2 \sin\phi_2$$

Note that at periapsis and apoapsis, ϕ = 90 degrees. Thus, letting *P1* and *P2* be these two points we get

(1.14) $$R_p V_p = R_a V_a$$

Let's now look at the energy of the above particle at points *P1* and *P2*. *Conservation of energy* states that the sum of the kinetic energy and the potential energy of a particle remains constant. The kinetic energy *T* of a particle is given by $mv^2/2$ while the potential energy of gravity *V* is calculated by the equation $-GMm/r$. Applying conservation of energy we have

$$T_1 + V_1 = T_2 + V_2, \quad \text{or}$$

$$\frac{m v_1^2}{2} - \frac{GMm}{r_1} = \frac{m v_2^2}{2} - \frac{GMm}{r_2}, \quad \text{or}$$

(1.15) $$v_2^2 - v_1^2 = 2GM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

From equations (1.14) and (1.15) we obtain

(1.16) $$Vp = \sqrt{\frac{2GMR_a}{R_p(R_a + R_p)}}, \quad \text{and}$$

(1.17) $$Va = \sqrt{\frac{2GMR_p}{R_a(R_a + R_p)}}$$

Rearranging terms we get

(1.18) $$R_a = \frac{R_p}{\left(\frac{2GM}{R_p V_p^2} - 1\right)}, \quad \text{and}$$

(1.19) $$R_p = \frac{R_a}{\left(\frac{2GM}{R_a V_a^2} - 1\right)}$$

Example 1.4: An artificial earth satellite is in an elliptical orbit which brings it to an altitude of 250 km at perigee and out to an altitude of 500 km at apogee. Calculate the velocity of the satellite at both perigee and apogee.

Given: Rp = (6,371 + 250) x 1,000 = 6,621,000 m
and Ra = (6,371 + 500) x 1,000 = 6,871,000 m

From equations (1.16) and (1.17),

Vp = SQRT[2 x GM x Ra / (Rp x (Ra + Rp))]
Vp = SQRT[2 x 3.986x10^14 x 6,871,000 / (6,621,000 x (6,871,000 + 6,621,000))]
Vp = 7,831 m/s

Va = SQRT[2 x GM x Rp / (Ra x (Ra + Rp))]
Va = SQRT[2 x 3.986x10^14 x 6,621,000 / (6,871,000 x (6,871,000 + 6,621,000))]
Va = 7,546 m/s

Example 1.5: A satellite in earth orbit passes through its perigee point at an altitude of 200 km above the earth's surface and at a velocity of 7,850 m/s. Calculate the apogee altitude of the satellite.

Given: Rp = (6,371 + 200) x 1,000 = 6,571,000 m and Vp = 7,850 m/s

From equation (1.18),

Ra = Rp / [2 x GM / (Rp x Vp^2) - 1]
Ra = 6,571,000 / [2 x 3.986x10^14 / (6,571,000 x 7,850^2) - 1]
Ra = 6,783,000 m

Altitude @ apogee = 6,783,000 / 1,000 - 6,371 = 412 km

The eccentricity *e* of an orbit is given by

(1.20) $$e = \frac{R_p V_p^2}{GM} - 1$$

Example 1.6: Calculate the eccentricity of the orbit for the satellite in problem 1.5.

Given: Rp = 6,571,000 m and Vp = 7,850 m/s

From equation (1.20),

e = Rp x Vp^2 / GM - 1
e = 6,571,000 x 7,850^2 / 3.986x10^14 - 1
e = 0.0159

If the semi-major axis *r* and the eccentricity *e* of an orbit are known, then the periapsis and apoapsis distances can be calculated by

(1.21) $R_p = r(1-e)$, and

(1.22) $R_a = r(1+e)$

also note, $R_p + R_a = 2r$

Example 1.7: A satellite in earth orbit has a semi-major axis of 6,700 km and an eccentricity of 0.01. Calculate the satellite's altitude at both perigee and apogee.

Given: r = 6,700 km and e = 0.01

From equations (1.21) and (1.22),

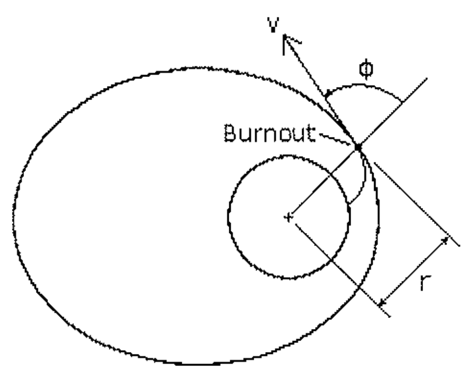
Rp = r x (1 - e)
Rp = 6,700 x (1 - .01)
Rp = 6,633 km

Altitude @ perigee = 6,633 - 6,371 = 262 km

Ra = r x (1 + e)
Ra = 6,700 x (1 + .01)
Ra = 6,767 km

Altitude @ apogee = 6,767 - 6,371 = 396 km

Launch of a Space Vehicle



The launch of a satellite or space vehicle consists of a period of powered flight during which the vehicle is lifted above the earth's atmosphere and accelerated to orbital velocity by a rocket, or launch vehicle. Powered flight concludes at burnout of the rocket's last stage at which time the vehicle begins its free flight. During free flight the space vehicle is assumed to be subjected only to the gravitational pull of the earth. If the vehicle moves far from the earth, its trajectory may be affected by the gravitational influence of the sun, moon, or another planet.

A space vehicle's orbit may be determined from the position and the velocity of the vehicle at the beginning of its free flight. A vehicle's position and velocity can be described by the variables r, v, and ϕ, where r is the vehicle's distance from the center of the earth, v is its velocity, and ϕ is the angle between the radius vector and the velocity vector (see figure above). If we let $r1$, $v1$, and ϕ_1 be the initial (launch) values of r, v, and ϕ, then we may consider these as given quantities. If we let point $P2$ represent the perigee, then equation (1.13) becomes

(1.23) $$v_2 = V_p = \frac{r_1 v_1 \sin\phi_1}{R_p}$$

Substituting equation (1.23) into (1.15), we can obtain an equation for the perigee radius Rp.

(1.24) $$\frac{r_1^2 v_1^2 \sin^2\phi_1}{R_p^2} - v_1^2 = 2GM\left(\frac{1}{R_p} - \frac{1}{r_1}\right)$$

Multiplying through by $-Rp^2/(r1^2 v1^2)$ and rearranging, we get

(1.25) $$\left(\frac{R_p}{r_1}\right)^2 (1-C) + \left(\frac{R_p}{r_1}\right) C - \sin^2\phi_1 = 0$$

$$\text{where } C = \frac{2GM}{r_1 v_1^2}$$

Note that this is a simple quadratic equation in the ratio $(Rp/r1)$ and that $2GM/(r1 \times v1^2)$ is a nondimensional parameter of the orbit.

Solving for *(Rp/r1)* gives

$$(1.26)\qquad \left(\frac{R_p}{r_1}\right)_{1,2} = \frac{-C \pm \sqrt{C^2 - 4(1-C)(-\sin^2\phi_1)}}{2(1-C)}$$

Like any quadratic, the above equation yields two answers. The smaller of the two answers corresponds to *Rp*, the periapsis radius. The other root corresponds to the apoapsis radius, *Ra*.

Example 1.8: A satellite is launched into earth orbit where its launch vehicle burns out at an altitude of 150 miles. At burnout the satellite's velocity is 26,000 ft/s with equal to 89 degrees. Calculate the satellite's altitude at perigee and apogee.

Given: r1 = (3,959 + 150) x 5,280 = 21,696,000 ft, v1 = 26,000 ft/s, and phi1=89 deg.

From equation (1.26),

(Rp / r1)1,2 = (-C +/- SQRT[C^2 - 4 x (1 - C) x -sin^2 phi1]) / (2 x (1 - C))

where C = 2 x GM / (r1 x v1^2)
C = 2 x 1.408x10^16 / (21,696,000 x 26,000^2)
C = 1.920

(Rp / r1)1,2 = (-1.920 +/- SQRT[1.920^2 - 4 x -0.920 x -sin^2(89)]) / (2 x -0.920)
(Rp / r1)1,2 = 0.9963 and 1.091

Perigee Radius, Rp = Rp1 = r1 x (Rp / r1)1

Rp = 21,696,000 x 0.9963
Rp = 21,616,000 ft

Altitude @ perigee = 21,616,000 / 5280 - 3,959 = 135 miles

Apogee Radius, Ra = Rp2 = r1 x (Rp / r1)2

Ra = 21,696,000 x 1.091
Ra = 23,670,000 ft

Altitude @ agogee = 23,670,000 / 5,280 - 3,959 = 524 miles

Please note, in practice launches are usually terminated at either perigee or apogee, i.e. ϕ=90. This condition results in the minimum use of fuel.

Equation (1.26) gives the values of *Rp* and *Ra* from which the eccentricity of the orbit can be calculated, however, it may be simpler to calculate the eccentricity *e* directly from the equation

$$(1.27)\qquad e = \sqrt{\left(\frac{r_1 v_1^2}{GM} - 1\right)^2 \sin^2\phi_1 + \cos^2\phi_1}$$

Example 1.9: Calculate the eccentricity of the orbit for the satellite in problem 1.8.

Given: r1 = 21,696,000 ft, v1 = 26,000 ft/s, and phi1= 89 deg.

From equation (1.27),

e = SQRT[(r1 x v1^2 / GM - 1)^2 x sin^2 (phi1) + cos^2 (phi1)]
e = SQRT[(21,696,000 x 26,000^2 / 1.408x10^16 - 1)^2 x sin^2(89) + cos^2(89)]
e = 0.0452

To pin down a satellite's orbit in space, we need to know the angle θ from the periapsis point to the launch point. This angle is given by

$$\tan\theta = \frac{\left(\frac{r_1 v_1^2}{GM}\right)\sin\phi_1\cos\phi_1}{\left(\frac{r_1 v_1^2}{GM}\right)\sin^2\phi_1 - 1} \quad (1.28)$$

Example 1.10: Calculate the angle from perigee point to launch point for the satellite in problem 1.8.

Given: r1 = 21,696,000 ft, v1 = 26,000 ft/s, and phi1 = 89 deg.

From equation (1.28),

tan(theta) = (r1 x v1^2 / GM) x sin(phi1) x cos(phi1) / [(r1 x v1^2 / GM) x sin^2(phi1) - 1]
tan(theta) = (21,696,000 x 26,000^2 / 1.408x10^16) x sin(89) x cos(89)
/ [(21,696,000 x 26,000^2 / 1.408x10^16) x sin^2(89) - 1]
tan(theta) = 0.440

theta = arctan(0.440)
theta = 23.7 deg.

Escape Velocity

We know that if we throw a ball up from the surface of the earth, it will rise for a while and then return. If we give it a larger initial velocity, it will rise higher and then return. There is a velocity, called the *escape velocity, Vesc*, such that if the ball is launched with an initial velocity greater than *Vesc*, it will rise and never return. We must give the particle enough kinetic energy to overcome all of the negative gravitational potential energy. Thus, if *m* is the mass of the ball, *M* is the mass of the earth, and *R* is the radius of the earth, the potential energy is *-GmM /R*. The kinetic energy of the ball, when it is launched, is $mv^2/2$. We thus have

$$\frac{GmM}{R} = \frac{mV_{esc}^2}{2}, \quad \text{or}$$

$$V_{esc} = \sqrt{\frac{2GM}{R}} \quad (1.29)$$

which is independent of the mass of the ball (or space vehicle).

Example 1.11: Calculate the escape velocity of a spacecraft launched from the surface of the earth. Likewise, calculate the escape velocity from the surface of the moon where the mass of the moon is 0.0123 times the mass of the earth and the moon's radius is 1,080 miles.

Part 1,

Given: R = 3,959 x 5,280 = 20,904,000 ft

From equation (1.29),

VESC = SQRT[2 x GM / R]
VESC = SQRT[2 x 1.408x10^16 / 20,904,000]
VESC = 36,700 ft/s

Part 2,

Given: R = 1,080 x 5,280 = 5,702,000 ft
GM = 1.408x10^16 x 0.0123 = 1.732x10^14

Eq. (1.29),

VESC = SQRT[2 x 1.732x10^14 / 5,702,000]
VESC = 7,794 ft/s

Thrust

Thrust is the force that propels a rocket or spacecraft. In this section we will take a look at how the application of thrust affects the orbit of a space vehicle. If you would like to know more about thrust and spacecraft propulsion, please refer to the section titled Rocket Propulsion.

A space vehicle in orbit experiences the sensation of weightlessness because the outward force of centrifugal acceleration perfectly balances the inward gravitational pull of the earth. By applying thrust, the space vehicle's velocity can be increased or decreased. If velocity is increased the outward centrifugal force also increases which "pulls" the vehicle to a higher orbit. Decreasing velocity lessens the centrifugal force and gravity "pulls" the vehicle to a lower orbit. Such altitude changes do not alter the inclination of the orbit, they merely reposition the vehicle within the same orbital plane. Applying thrust at right angles to the orbital plane modifies the inclination. These maneuvers, called *plane changes*, burn considerably more propellant than altitude changes.

For a spacecraft to perform an altitude change, two engine burns are required. To change to a higher orbit, the spacecraft fires its engine to increase velocity, thus placing it in an elliptical orbit with an apoapsis equal to the new altitude. When the spacecraft reaches apoapsis, a second burn is performed to once again increase velocity, thereby placing the vehicle in a circular orbit. For a spacecraft to change to a lower orbit, the procedure is reversed. The craft fires its engine in the direction of travel to decrease velocity, thus dropping the spacecraft into an elliptical orbit with a periapsis equal to the new altitude. When reaching periapsis the engine is fired to decrease velocity further, thereby circularizing the orbit.

When propulsive maneuvers are used to alter the orbit of a space vehicle, engineers calculate the magnitude of the velocity change required to achieve the desired alteration. This change in velocity is called *delta v* (Δv). For altitude changes, Δv can be calculated from equations (1.16) and (1.17).

Example 1.12: A spacecraft is in a circular earth orbit with an altitude of 150 miles. Calculate the delta v's required to change to a circular orbit with an altitude of 500 miles.

Initial orbit,

Given: r1 = (3,959 + 150) x 5,280 = 21,696,000 ft

From equation (1.6),

v1 = SQRT[GM / r1]
v1 = SQRT[1.408x10^16 / 21,696,000]
v1 = 25,470 ft/s

Final orbit,

Given: r2 = (3,959 + 500) x 5,280 = 23,544,000 ft

Eq. (1.6),

v2 = SQRT[GM / r2]
v2 = SQRT[1.408x10^16 / 23,544,000]
v2 = 24,450 ft/s

Transfer orbit,

Given: Rp = r1 = 21,696,000 ft
Ra = r2 = 23,544,000 ft

From equations (1.16) and (1.17),
Vp = SQRT[2 x GM x Ra / (Rp x (Ra + Rp))]
Vp = SQRT[2 x 1.408x10^16 x 23,544,000 / (21,696,000 x (23,544,000 + 21,696,000))]
Vp = 25,990 ft/s

Va = SQRT[2 x GM x Rp / (Ra x (Ra + Rp))]
Va = SQRT[2 x 1.408x10^16 x 21,696,000 / (23,544,000 x (23,544,000 + 21,696,000))]
Va = 23,950 ft/s

delta v, 1st burn = Vp - v1

delta v = 25,990 - 25,470
delta v = 520 ft/s

delta v, 2nd burn = v2 - Va

delta v = 24,450 - 23,950
delta v = 500 ft/s

Drag

Drag is the resistance offered by a gas or liquid to a body moving through it. A spacecraft is subjected to drag forces when moving through a planet's atmosphere. This drag is greatest during launch and reentry, however, even a space vehicle in low earth orbit experiences some drag as it moves through the earth's tenuous upper atmosphere. In time, the action of air drag on a space vehicle will cause it to spiral back into the atmosphere, eventually to disintegrate or burn up. If a space vehicle comes within 80 to 100 miles of the earth's surface, air drag will bring it down in a few days, with final disintegration occurring at an altitude of about 50 miles. This deterioration of a spacecraft's orbit is called *decay*.

The drag F_D on a body is calculated by the equation

$$F_D = C_D \rho \left(\frac{v^2}{2}\right) A \qquad (1.30)$$

where C_D is the drag coefficient, ρ is the air density, v is the body's velocity, and A is the area of the body normal to the flow.

The drag coefficient is dependent on the geometric form of the body and is generally determined by experiment. With most bodies the drag coefficient tends to increase drastically at a Mach number of about 0.70. This is because at some place in the flow field supersonic flow is occurring resulting in the formation of a shock wave. As the Mach number increases beyond about 2, for most bodies there is a drop in the valve of the drag coefficient. In supersonic flow the best nose form is a sharp point. This tends to minimize the effect of the shock wave. Drag coefficients for some common shapes are given by the appendix Drag coefficients as a function of Mach number. Mach number is given by

(1.31) $$N_m = \frac{v}{c}$$

where *c* is the acoustic velocity and is calculated by the equation

(1.32) $$c = \sqrt{kRT}$$

where *k* is the specific heat ratio equal to 1.40, *R* is the gas constant equal to 1,715 ft-lb/slug-R in U.S. units and 287 N-m/kg-K in S.I. units, and *T* is the air temperature.

Air temperature *T* and density ρ are given by the following appendix:

Physical Properties of Standard Atmosphere in U.S. units or S.I. Units.

For a space vehicle, the area *A* depends on the *angle of attack*, which is the angle between the velocity vector and the longitudinal axis of the space vehicle. If the angle of attack is zero, then *A* equals the cross-sectional area of the spacecraft.

Example 1.13: A Saturn V launch vehicle's first stage cuts off at an altitude of 38 miles and a velocity of 6,100 mph. Calculate the air drag on the vehicle at burnout. Assume the vehicle's maximum diameter is 33 feet and the angle of attack is zero.

Given: Altitude = 38 x 5280 = 200,640 ft, v = 6,100 x 5,280 / 3,600 = 8,950 ft/s, and A = pi x (33 / 2)^2 = 855 ft^2

From the physical properties of the standard atmosphere,

T = 457 deg R
rho = 5.270x10^-7 slug/ft^3

From equations (1.32) and (1.31),

c = SQRT[k x R x T]
c = SQRT[1.40 x 1,715 x 457]
c = 1,050 ft/s

Nm = v / c
Nm = 8,950 / 1,050
Nm = 8.5

From drag coefficients as a function of Mach number,

CD = 0.25 (assuming a sharp-nosed projectile)

Then using equation (1.30),

FD = CD x rho x (v^2 / 2) x A
FD = 0.25 x 5.270x10^-7 x (8,950^2 / 2) x 855
FD = 4,510 lb