Problem 1. Let \( \phi \in C^\infty_c(\mathbb{R}^n) \) with \( \int \phi(x) \, dx = 1, \phi \geq 0, \text{supp } \phi \subset B_1(0) \), and let \( \delta_j \to 0 \).

For \( y \in \mathbb{R}^n \), let \( f_{y,j}(x) = \delta_j^{-n} \phi((x-y)/\delta_j) \).

(i) Show that the distribution \( u_{y,j} = \iota_{f_{y,j}} \) given by \( f_{y,j} \) converges to \( \delta_y \) as \( j \to \infty \), i.e.
\[
\lim_{j \to \infty} \int_{\mathbb{R}^n} f_{y,j}(x) \psi(x) \, dx = \psi(y).
\]

(ii) Show that even if \( \psi \) is merely a continuous function of compact support, the function
\[
\psi_j(y) = \int_{\mathbb{R}^n} f_{y,j}(x) \psi(x) \, dx = \int_{\mathbb{R}^n} \delta_j^{-n} \phi((x-y)/\delta_j) \psi(x) \, dx
\]

is actually a \( C^\infty \) function of compact support.

(iii) Show that if \( \psi \) is a continuous function of compact support, then \( \psi_j \to \psi \) uniformly, i.e.
\[
\sup_{y \in \mathbb{R}^n} |\psi_j(y) - \psi(y)| \to 0
\]
as \( j \to \infty \).

Hint: \( \psi \) continuous of compact support implies that \( \psi \) is uniformly continuous, i.e. for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |\psi(x) - \psi(y)| < \epsilon \) if \( |x-y| < \delta \).

Note: This shows that continuous functions of compact support can be approximated, in the natural norm on continuous functions, by \( C^\infty \) functions of compact support.

Problem 2. Consider the conservation law
\[
u_t + (f(u))_x = 0, \quad u(x,0) = \phi(x),
\]
with \( f \in C^2(\mathbb{R}) \). Let \( v = f'(u) \). Show that if \( f'' \neq 0 \) and \( u \) is continuous and is \( C^1 \) apart from jump discontinuities in its first derivatives then \( v \) has the same properties and satisfies Burgers’ equation. (Note that the Rankine-Hugoniot condition is vacuous: there are no shocks.) If \( f \) is strictly convex, i.e. \( f'' > 0 \), conclude that one can reduce general scalar conservation laws to Burgers’ equation, i.e. that one can find \( u \) by solving for \( v \) first.

Is the same true if \( u \) has jump discontinuities, i.e. is it true that if \( u \) satisfies the Rankine-Hugoniot conditions then \( v \) does as well?

Problem 3. Consider the PDE
\[
u_{tt} - \nabla \cdot (c^2 \nabla u) + qu = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),
\]
where \( c, q \geq 0 \), depend on \( x \) only, and \( c \) is bounded between positive constants, i.e. for some \( c_1, c_2 > 0, c_1 \leq c(x) \leq c_2 \) for all \( x \in \mathbb{R}^n \). Assume that \( u \) is \( C^2 \) throughout this problem, and \( u \) is real-valued. (All calculations would go through if one wrote \( |u_t|^2 \), etc., in the complex valued case.)

(i) Fix \( x_0 \in \mathbb{R}^n \) and \( R_0 > 0 \), and for \( t < \frac{R_0}{c_2} \), let
\[
E(t) = \int_{|x-x_0|<R_0-c_2t} (u_t^2 + c(x)^2 |\nabla u|^2 + q(x)u^2) \, dx.
\]

Show that \( E \) is decreasing with \( t \) (i.e. non-increasing). (Hint: to make sure you don’t forget anything in the calculation, do it first on the line, when \( n = 1 \).)
(ii) Suppose that \( \text{supp } \phi, \text{supp } \psi \subset \{ |x| \leq R \} \), i.e. are 0 outside this ball. Show that \( u(x, t) = 0 \) if \( t \geq 0, |x| > R + c_2 t \), i.e. the wave indeed propagates at speed \( \leq c_2 \).

(iii) Show that there is at most one real-valued \( C^2 \) solution of (1).

**Problem 4.** Consider the wave equation on \( \mathbb{R}^n \):

\[
 u_{tt} - c^2 \Delta u = f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),
\]

and write \( x = (x', x_n) \) where \( x' = (x_1, \ldots, x_{n-1}) \)

(i) Show that if

\[
 f(x', x_n, t) = f(x', -x_n, t), \quad \phi(x', x_n) = \phi(x', -x_n), \quad \psi(x', x_n) = \psi(x', -x_n)
\]

for all \( x \) and \( t \), i.e. if \( f, \phi, \psi \) are all even functions of \( x_n \), then \( u \) is an even function of \( x_n \) as well. (Hint: Consider \( u(x', x_n, t) - u(x', -x_n, t) \), show that it solves the homogeneous wave equation with 0 initial conditions.)

(ii) Show that if

\[
 f(x', x_n, t) = -f(x', -x_n, t), \quad \phi(x', -x_n) = -\phi(x', x_n), \quad \psi(x', x_n) = -\psi(x', x_n)
\]

for all \( x \) and \( t \), i.e. if \( f, \phi, \psi \) are all odd functions of \( x_n \), then \( u \) is an odd function of \( x_n \) as well.

(iii) If \( u \) is continuous, and is an odd function of \( x_n \), show that \( u(x', 0, t) = 0 \) for all \( x' \) and \( t \).

(iv) If \( u \) is a \( C^1 \) and is an even function of \( x_n \), show that \( \partial_{x_n} u(x', 0, t) = 0 \) for all \( x' \) and \( t \).

These facts will enable us to solve the wave equation in the half space \( x_n > 0 \) with Dirichlet or Neumann boundary conditions later in the course.

**Problem 5.** Use the maximum principle for Laplace’s equation on \( \mathbb{R}^n \) to show the following statement: Suppose that \( u \in C^2(\mathbb{R}^n) \) and \( \Delta u = 0 \). Suppose moreover that \( u(x) \to 0 \) at infinity uniformly in the following sense:

\[
 \sup_{|x| > R} |u(x)| \to 0
\]

as \( R \to \infty \). Then \( u(x) = 0 \) for all \( x \in \mathbb{R}^n \). (Hint: Apply the maximum principle shown in class for the ball \( \Omega = \{ x : |x| < R \} \) and for both \( u \) and \(-u\).)

Use this to show that the solution of Laplace’s equation on \( \mathbb{R}^n \):

\[
 \Delta u = f,
\]

with \( f \) given, is unique in the class of functions \( u \) such that \( u \in C^2(\mathbb{R}^n) \) and \( u(x) \to 0 \) at infinity uniformly.

**Problem 6.** Suppose \( \Omega \) is a bounded \( C^1 \) domain and \( A(x) = (A_{ij}(x))_{i,j=1}^n \) is symmetric, positive definite in the sense that there is \( c_0 > 0 \) such that \( A(x)v \cdot v \geq c_0 |v|^2 \) for all \( v \in \mathbb{R}^n \), \( x \in \bar{\Omega} \), and \( q \geq 0 \), with \( A_{ij} \) being \( C^1 \), \( q \) continuous on \( \bar{\Omega} \). Show that there is \( C > 0 \) such that real valued solutions \( u \in C^2(\bar{\Omega}) \) of the PDE

\[
 \nabla \cdot (A(x)\nabla u) - qu = f
\]

with Dirichlet boundary conditions \( u|_{\partial \Omega} = 0 \) satisfy

\[
 \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \leq C \int_{\Omega} f^2 \, dx.
\]