Problem 1. Solve the inhomogeneous heat equation on the half-line for Dirichlet boundary conditions:

\[ u_t - ku_{xx} = f, \quad u(x,0) = \phi(x), \quad u(0,t) = 0, \]

in two different ways:

(i) Using Duhamel’s principle, applied on the half line directly, and the solution formula for the homogeneous equation derived in class/lecture notes (i.e. with \( f = 0 \)) on the half line.

(ii) Using the appropriate extension of \( f \) and \( \phi \) to the whole real line and solving the inhomogeneous PDE on the real line.

Problem 2. Derive Duhamel’s principle for the wave equation on \( \mathbb{R} \)

\[ u_{tt} - c^2 \partial_x^2 u = f, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \]

by setting up a first order system for \( U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad v = u_t, \) namely

\[
\begin{align*}
u_t - v &= 0, \quad u(x,0) = \phi(x), \\
v_t - c^2 \partial_x^2 u &= f, \quad v(x,0) = \psi(x).
\end{align*}
\]

Thus, one has

\[ \partial_t U - AU = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad U(0,x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \]

where

\[ A = \begin{bmatrix} 0 & \text{Id} \\ c^2 \partial_x^2 & 0 \end{bmatrix}. \]

This is now a first order equation in time, so Duhamel’s principle for first order equations is applicable, and gives the solution of the inhomogeneous equation as

\[ U(x,t) = \mathcal{S}(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix}(x) + \int_0^t \mathcal{S}(t-s) \begin{bmatrix} 0 \\ f_s \end{bmatrix}(x) \, ds, \]

where \( \mathcal{S} \) is the solution operator for the homogeneous problem \( \partial_t U - AU = 0 \). You need to work this out explicitly, in particular what \( \mathcal{S} \) is, to derive the solution of the wave equation.

Problem 3. (i) Consider the following eigenvalue problem on \([0, \ell] \):

\[ -X'' = \lambda X, \quad X(0) = 0, \quad X'(\ell) = 0. \]

Find all eigenvalues and eigenfunctions.

(ii) Using separation of variables, find the general ‘separated’ solution of the wave equation

\[ u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0. \]

(iii) Solve the wave equation with initial conditions

\[ u(x,0) = \sin(3\pi x/(2\ell)) - 2 \sin(5\pi x/(2\ell)), \quad u_t(x,0) = 0. \]

(iv) Using separation of variables, find the general ‘separated’ solution of the heat equation

\[ u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0, \]

here \( k > 0 \) constant.
Problem 4. Consider the wave equation on a ring of length $2\ell$. We let $x$ be the arclength variable along the ring, $x \in [-\ell, \ell]$. We would like to understand wave propagation along the ring, so consider the wave equation with periodic boundary conditions:

$$u_{tt} = c^2 u_{xx}, \quad u(-\ell, t) = u(\ell, t), \quad u_x(-\ell, t) = u_x(\ell, t).$$

(i) Find the general ‘separated’ solution.

(ii) Find the solution with initial condition $u(x, 0) = 0$, $u_t(x, 0) = \cos(2\pi x/\ell) - \sin(\pi x/\ell)$, $x \in [-\ell, \ell]$.

(iii) Give an alternative method of solution by extending $u$ to be a $2\ell$-periodic function in $x$ on all of $\mathbb{R}$, and using d’Alembert’s formula.

(iv) How do singularities of $u$ propagate? That is, if the only singularity of the initial data is at some $x_0$ (i.e. they are $C^\infty$ elsewhere), where can $u$ be singular? Interpret this physically.

Problem 5. The goal of this problem is to show that if $u \in D'(\mathbb{R}^3)$ and $\Delta u = f$ satisfies $x_0 \notin \text{singsupp } f$, i.e. $f$ is $C^\infty$ near $x_0$, then $u$ is $C^\infty$ near $x_0$. This is called elliptic regularity: $\Delta$ is elliptic, and for an elliptic operator $P$ if $Pu$ is $C^\infty$ near some $x_0$ then so is $u$.

We achieve this as follows.

(i) First suppose that $u$ is a $C^2$ function. Let $\phi \in C^\infty_c(\mathbb{R}^3)$ be identically 1 near $x_0$ such that $f$ is $C^\infty$ on supp $\phi$. Then show that $\Delta(\phi u) = \phi \Delta u + v$, where $v$ is a compactly supported distribution that vanishes near $x_0$. Now as $w = \phi u$ is compactly supported,

$$w(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|} \Delta_y(\phi(y)u(y)) \, dy.$$

(ii) Expand $\Delta_y(\phi(y)u(y))$ as above. To analyze

$$\int_{\mathbb{R}^3} \frac{1}{4\pi |x-y|} v(y) \, dy$$

for $x$ near $x_0$, note that if $x$ is near $x_0$ and $y \in \text{supp } v$ then $x \neq y$, so $|x-y|^{-1}$ is $C^\infty$. On the other hand, $\phi \Delta u$ is $C^\infty$ by assumption. Write the corresponding part of the convolution as

$$\int_{\mathbb{R}^3} \frac{1}{4\pi |y|} \phi(x-y)(\Delta u)(x-y) \, dy,$$

and deduce that it is $C^\infty$.

(iii) Suppose now that $u \in D'$. Proceed as above, writing

$$w = -\frac{1}{4\pi |x|} * (\Delta(\phi u)),$$

convolution in the sense of distributions (so $w$ is merely a distribution), and show that both parts are $C^\infty$ near $x_0$. You do not have to be very careful in writing up this part; there are some technicalities, but the point is to get the main idea.