MATH 173: PROBLEM SET 9
DUE 10AM, WEDNESDAY, MARCH 9, 2016

Problem 1. (i) Using the general 'separated' solution you found in Problem Set 8, Problem 3, solve the wave equation

$$u_{tt} = c^2 u_{xx},\ u(0,t) = 0,\ u_x(\ell,t) = 0,$$

with initial conditions

$$u(x,0) = 0,\ u_t(x,0) = x(x-\ell)^2.$$

(ii) Using the general 'separated' solution you found in Problem 3, solve the heat equation

$$u_t = k u_{xx},\ u(0,t) = 0,\ u_x(\ell,t) = 0,\ u(x,0) = x(x-\ell)^2.$$

You may assume throughout that the generalized Fourier series you construct converge to the function they are supposed to represent.

Problem 2. Let $\phi(x) = |x|$ on $[-\pi,\pi]$. Let

$$f(x) = a_0 + a_1\cos x + a_2\cos(2x) + b_1\sin x + b_2\sin(2x),$$

With what choice of the coefficients a_j and b_j is the L^2 error $\|f-\phi\|$ minimal? (Here $\|f-\phi\|^2 = \int_{-\pi}^{\pi} |f(x)-\phi(x)|^2\,dx$.)

Problem 3. Let $V = C([0,\ell])$ with the inner product $\langle f,g\rangle = \int_0^\ell f(x)\,\overline{g(x)}\,dx$. If $D \subset V$ is a subspace, and $A : D \to V$ is symmetric, we say that A is positive if $\langle Av, v\rangle \geq 0$ for all $v \in D$.

(i) Show that if A is positive than all eigenvalues λ of A are ≥ 0.

(ii) Show that $A = -\frac{d^2}{dx^2}$ with domain

$$D = \{f \in C^2([0,\ell]) :\ f(0) = f'(\ell) = 0\}$$

is symmetric and is positive.

(iii) Show that $A = \frac{d^4}{dx^4}$ with domain

$$D = \{f \in C^4([0,\ell]) :\ f(0) = f'(0) = f(\ell) = f'(\ell) = 0\}$$

is symmetric and is positive.

Problem 4. Let $\phi(x) = x(\ell - x)$ on $[0,\ell]$.

(i) Find the Fourier sine series of ϕ, and state what it converges to for $x \in \mathbb{R}$.

(ii) Find the Fourier cosine series of ϕ, and state what it converges to for $x \in \mathbb{R}$.

(iii) Compare the decay rates of the coefficients of the two series as $n \to \infty$. Why do the coeffients decay faster in one of the cases?

Problem 5. For both of the following functions discuss whether the Fourier sine series converges uniformly or in L^2:

(i) $\phi(x) = x$ on $[0,\ell]$,

(ii) $\phi(x) = x(\ell - x)^2$ on $[0,\ell]$,

Justify your answer by quoting the relevant convergence theorems. You do not need to compute the respective Fourier series.