Problem 1.  

(i) Using the general ‘separated’ solution you found in Problem Set 8, Problem 3, solve the wave equation

\[ u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u_x(\ell, t) = 0, \]

with initial conditions

\[ u(x, 0) = 0, \quad u_t(x, 0) = x(x - \ell)^2. \]

(ii) Using the general ‘separated’ solution you found in Problem 3, solve the heat equation

\[ u_t = ku_{xx}, \quad u(0, t) = 0, \quad u_x(\ell, t) = 0, \quad u(x, 0) = x(x - \ell)^2. \]

You may assume throughout that the generalized Fourier series you construct converge to the function they are supposed to represent.

Problem 2. Let \( \phi(x) = |x| \) on \([−\pi, \pi]\). Let

\[ f(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + b_1 \sin x + b_2 \sin(2x), \]

With what choice of the coefficients \( a_j \) and \( b_j \) is the \( L^2 \) error \( ∥f - \phi∥ \) minimal? (Here \( ∥f - \phi∥^2 = \int_{−\pi}^{\pi} |f(x) - \phi(x)|^2 \, dx \).)

Problem 3. Let \( V = C([0, \ell]) \) with the inner product \( \langle f, g \rangle = \int_0^\ell f(x) \overline{g(x)} \, dx \). If \( D \subset V \) is a subspace, and \( A : D \to V \) is symmetric, we say that \( A \) is positive if

\[ \langle Av, v \rangle \geq 0 \text{ for all } v \in D. \]

(i) Show that if \( A \) is positive than all eigenvalues \( \lambda \) of \( A \) are \( \geq 0 \).

(ii) Show that \( A = -\frac{d^2}{dx^2} \) with domain

\[ D = \{ f \in C^2([0, \ell]) : f(0) = f'(0) = 0 \} \]

is symmetric and is positive.

(iii) Show that \( A = \frac{d^4}{dx^4} \) with domain

\[ D = \{ f \in C^4([0, \ell]) : f(0) = f'(0) = f(\ell) = f'(\ell) = 0 \} \]

is symmetric and is positive.

Problem 4. Let \( \phi(x) = x(\ell - x) \) on \([0, \ell]\).

(i) Find the Fourier sine series of \( \phi \), and state what it converges to for \( x \in \mathbb{R} \).

(ii) Find the Fourier cosine series of \( \phi \), and state what it converges to for \( x \in \mathbb{R} \).

(iii) Compare the decay rates of the coefficients of the two series as \( n \to \infty \). Why do the coeffients decay faster in one of the cases?

Problem 5. For both of the following functions discuss whether the Fourier sine series converges uniformly or in \( L^2 \):

(i) \( \phi(x) = x \) on \([0, \ell]\),

(ii) \( \phi(x) = x(\ell - x)^2 \) on \([0, \ell]\),

Justify your answer by quoting the relevant convergence theorems. You do not need to compute the respective Fourier series.