Problem 1. Let $\psi \in C(\mathbb{R})$ be given by:

$$
\psi(x) = \begin{cases} 
0, & x < -1, \\
1 + x, & -1 < x < 0, \\
1 - x, & 0 < x < 1, \\
0, & x > 1,
\end{cases} \quad (1)
$$

so that it verifies $\psi \geq 0$, $\psi(x) = 0$ if $|x| \geq 1$ and $\int_{\mathbb{R}} \psi(x)dx = 1$.

Consider $(\psi_j)_{j \geq 1}$ constructed as $\psi_j(x) = j \psi(jx)$, so that $\psi_j(x) = 0$ if $|x| \geq 1/j$, and $\int_{\mathbb{R}} \psi_j(x)dx = 1$, for all $j \geq 1$.

(1) Let us show that $\iota_{\psi_j} \rightarrow \delta_0$ in $D'(\mathbb{R})$. By definition, it means that we need to prove that, for all $\phi \in C^\infty_c(\mathbb{R})$, $\int_{\mathbb{R}} \psi_j(x)\phi(x)dx \rightarrow \delta_0(\phi) = \phi(0)$. So let’s prove that for every $\epsilon > 0$, there exists $j_0$ such that for all $j \geq j_0$,

$$
\left| \int_{\mathbb{R}} \psi_j(x)\phi(x)dx - \phi(0) \right| < \epsilon. \quad (2)
$$

Let $\epsilon > 0$ and $\phi \in C^\infty_c(\mathbb{R})$. Since $\phi$ is continuous, there exists $\eta > 0$ such that for all $|x| < \eta$, $|\phi(x) - \phi(0)| < \epsilon$. Then consider $j_0$ such that $1/j_0 < \eta$. Since we know that $\psi_j(x) = 0$ for all $|x| \geq 1/j$, $\int_{\mathbb{R}} \psi_j(x)dx = 1$, and $\psi_j(x) \geq 0$ for all $x$, for all $j \geq j_0$ we have:

$$
\left| \int_{\mathbb{R}} \psi_j(x)\phi(x)dx - \phi(0) \right| = \int_{\mathbb{R}} \psi_j(x)\phi(x)dx - \int_{\mathbb{R}} \psi_j(x)\phi(0)dx \\
\leq \int_{-1/j}^{1/j} \psi_j(x)|\phi(x) - \phi(0)|dx \\
\leq \epsilon \int_{-1/j}^{1/j} \psi_j(x)dx = \epsilon. \quad (3)
$$

(2) Let $\phi \in C^\infty_c(\mathbb{R})$ be such that $\phi(x) = 1$ for all $|x| < 1$. Then, for $j \geq 1$,

$$
\int_{\mathbb{R}} \psi_j(x)^2 \phi(x)dx = \int_{-1/j}^{0} j^2(1 + jx)^2dx + \int_{1/j}^{1} j^2(1 - jx)^2dx = \frac{2j}{3}. \quad (4)
$$
Therefore, as \( j \to +\infty \), \( \int \psi_j(x)^2 \phi(x) dx \to +\infty \), and \( \left\{ t_{\psi_j}(\phi) \right\}_{j=1}^{+\infty} \) does not converge. Consequently, \( \left\{ t_{\psi_j}(\phi) \right\}_{j=1}^{+\infty} \) does not converge to any distribution since \( \left\{ t_{\psi_j}(\phi) \right\}_{j=1}^{+\infty} \) does not converge for the very \( \phi \) we exhibited.

(3) We have just shown that \( \iota_{\psi_j} \to \delta_0 \), but \( \left\{ \iota_{\psi_j} \phi \right\}_{j=1}^{+\infty} \) does not converge to any distribution. Therefore there is no continuous extension of the map \( Q : f \mapsto f^2 \) on \( C(\mathbb{R}) \) to \( D'(\mathbb{R}) \).

Problem 2. In \( t \geq 0 \), we consider the conservation law
\[
\frac{du}{dt} + f(u)u_x = 0, \quad u(x, 0) = \phi(x),
\]
and suppose that \( u \) is \( C^1 \) except that it has a jump discontinuity along a \( C^1 \)-curve given by \( x = \xi(t) \).

Let us show that \( u \) is a weak solution of the PDE (5) if and only if \( u_+ \) and \( u_- \) solve the PDE in the classical sense. First, let’s suppose that \( u \) is a weak solution. Then, for all \( \psi \in C_\infty^\infty(\mathbb{R} \times [0, +\infty)) \)
\[
0 = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (u \psi_t + f(u) \psi_x) \, dx \, dt + \int_{-\infty}^{+\infty} \phi(x) \psi(0, x) \, dx = 0. \tag{6}
\]

Now let choose \( \psi \) such that \( \psi(x, 0) = 0 \), and break up the first integral into the regions \( \Omega_+ \) and \( \Omega_- \). We then get
\[
0 = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (u \psi_t + f(u) \psi_x) \, dx \, dt
= \int_{\Omega_-} (u_- \psi_t + f(u_-) \psi_x) \, dx \, dt + \int_{\Omega_+} (u_+ \psi_t + f(u_+) \psi_x) \, dx \, dt
\]
\[
= \int_{\Omega_+} (u_+ \psi_t + f(u_+) \psi_x) \, dx \, dt
= -\int_{\Omega_-} ((u_-) \psi_t + (f(u_-)) \psi_x) \, dx \, dt
\]
\[
= \pm \int_{x=\xi(t)} (u_\pm \psi \nu_2 + f(u_\pm) \psi \nu_1) \, ds, \tag{7}
\]
where \( \nu = (\nu_1, \nu_2) \) is the outward unit normal to \( \Omega_- \) (so also the opposite of the outward normal to \( \Omega_+ \)).

Since \( u \) is a weak solution, and \( u \) is smooth on either side of the curve \( x = \xi(t) \), \( u_\pm \) is a strong solution of (5) in \( \Omega_\pm \) and
\[ \int \int_{\Omega_x} \left( (u_\pm)_t + (f(u_\pm))_x \right) \psi \, dx \, dt = 0. \]  

(9)

Therefore, for all smooth function \( \psi \), we have

\[ \int_{x=\xi(t)} (u_- \nu_2 + f(u_-) \nu_1) \, ds - \int_{x=\xi(t)} (u_+ \nu_2 + f(u_+) \nu_1) \, ds = 0, \]  

(10)

which leads to

\[ u_-(\xi(t), t) \nu_2 + f(u_-(\xi(t), t)) \nu_1 = u_+(\xi(t), t) \nu_2 + f(u_+(\xi(t), t)) \nu_1, \]  

(11)

and

\[ f(u_-(\xi(t), t)) - f(u_+(\xi(t), t)) = -\frac{\nu_2}{\nu_1} (u_-(\xi(t), t) - u_+(\xi(t), t)) = \xi'(t)(u_-(\xi(t), t) - u_+(\xi(t), t)), \]  

(12)

which is exactly the Rankine-Hugoniot jump condition.

Hence \( u_+ \) and \( u_- \) solve the PDE in the classical sense.

Conversely, let’s suppose that \( u_+ \) and \( u_- \) solve the PDE in the classical sense. Then (9) and (12) hold. By performing the previous operations in reverse order and using similar arguments, we get that \( u \) is indeed a weak solution.

Now let’s look at the Burgers’ equation with particular initial condition:

\[
\begin{cases}
  u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\
  u(x, 0) = \phi(x) = \begin{cases}
    1, \quad x < 0 \\
    0, \quad x > 0
  \end{cases}
\end{cases}
\]  

(13)

and let’s find a weak solution using the previous analysis.

From the method of characteristics for first-order quasi-linear PDEs, we know that we have an implicit solution where \( u \) is smooth, namely \( u = \phi(x - ut) \).

Moreover \( u \) is constant along the projected characteristic curves given by \( x_r(t) = \phi(r)t + r \).

- If \( r < 0 \), then \( \phi(r) = 1 \), which implies that the characteristic curves are
  \[ x_r(t) = t + r, \quad r < 0, \]  
  (14)
  and the solution \( u = u_- \) should equal 1 along those curves,

- If \( r > 0 \), then \( \phi(r) = 0 \), which implies that the characteristic curves are
  \[ x_r(t) = r, \quad r > 0, \]  
  (15)
  and the solution \( u = u_+ \) should equal 0 along those curves.
Furthermore, at the jump, we have the Rankine-Hugoniot jump condition:

\[
\frac{(u_-)^2}{2} - \frac{(u_+)^2}{2} = \xi'(t)(u_- - u_+),
\]

which reduces to

\[
\xi'(t) = \frac{1}{2}.
\]

Moreover, the curve \( x = \xi(t) \) contains the point \((x, t) = (0, 0)\) (we indeed have a discontinuity for the initial function at \( x = 0 \)). Therefore the curve of discontinuity is given by \( x = \frac{t}{2} \) and the weak solution is:

\[
u(x, t) = \begin{cases} 
1, & x < \frac{t}{2} \\
0, & x > \frac{t}{2}
\end{cases}
\]

which means that the discontinuity (shock) is moving at speed \(1/2\).

**Problem 3.** Consider the Burgers’ equation

\[
\begin{cases}
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} = 0, & x \in \mathbb{R}, \; t > 0, \\
u(x, 0) = \phi(x) = \begin{cases} 
0, & x < 0 \\
\frac{x}{\epsilon}, & 0 < x < \epsilon \\
1, & x > \epsilon
\end{cases}
\end{cases}
\]

Let’s solve the equation. Once again, we know that \( u \) is constant along the projected characteristic curves \( x_r(t) = \phi(t)t + r \).

- If \( r < 0 \), then \( \phi(r) = 0 \), which implies that the characteristic curves are

\[
x_r(t) = r, \; r < 0,
\]

and the solution \( u(x, t) = 0 \) along those curves,

- If \( 0 < r < \epsilon \), then \( \phi(r) = \frac{r}{\epsilon} \), which implies that the characteristic curves are

\[
x_r(t) = \frac{r}{\epsilon} + r, \; 0 < r < \epsilon,
\]

and the solution \( u(x, t) = \frac{r}{\epsilon} = \frac{x}{t + \epsilon} \) along those curves,

- If \( r > \epsilon \), then \( \phi(r) = 1 \), which implies that the characteristic curves are

\[
x_r(t) = t + r, \; r > \epsilon,
\]

and the solution \( u(x, t) = 1 \) along those curves.
For \( t \geq 0 \), the characteristic curves do not intersect one another and the solution is then defined as

\[
u(x,t) = \begin{cases} 
0, & x < 0 \\
\frac{x}{t + \epsilon}, & 0 < x < t + \epsilon \\
1, & x > t + \epsilon
\end{cases}
\]

(23)

Now as \( \epsilon \to 0 \), it converges to

\[
v(x,t) = \begin{cases} 
0, & x < 0 \\
\frac{x}{t}, & 0 < x < t \\
1, & x > t
\end{cases}
\]

(24)

\( v \) is indeed a weak solution of the Burgers’ equation with \( \phi(x) = H(x) \). To verify this, we use the characterization given in Problem 4, meaning that we consider here the domains \( \Omega_1 = x < 0 \), \( \Omega_2 = 0 < x < t \) and \( \Omega_3 = x > t \), corresponding to the domains where \( v \) is \( C^1 \). We easily verify that the PDE holds pointwise in \( \Omega_1 \cup \Omega_2 \cup \Omega_3 \), that the initial condition holds at \( t = 0 \) and that we have the Rankine-Hugoniot jump condition at each interface (\( v \) is continuous here, so there is nothing to verify).

If we are looking for another weak solution which is piecewise constant, using the same procedure as in Problem 4, we get another solution given by

\[
w(x,t) = \begin{cases} 
0, & x < \frac{t}{2} \\
\frac{t}{2}, & \frac{t}{2} < x < t \\
1, & x > t
\end{cases}
\]

(25)

However, \( w \) does not satisfy the entropy condition since

\[
\begin{align*}
\forall (w_-) = w_- = 0, \\
\text{and} \\
\forall (w_+)^{'} = w_+ = 1, \\
\text{but} \\
\frac{f(w_-) - f(w_+)}{w_- - w_+} = \frac{1}{2},
\end{align*}
\]

(26)

while the solution \( u \) of Problem 4. satisfies it since

\[
\begin{align*}
\forall (u_-) = u_- = 1, \\
\text{and} \\
\forall (u_+)^{'} = u_+ = 0, \\
\text{while} \\
\frac{f(w_-) - f(w_+)}{w_- - w_+} = \frac{1}{2}.
\end{align*}
\]

(27)
Problem 4.

(1) \[ u_{xx} - u_{xy} - 2u_{yy} = 0. \] (28)

We have \( A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & -2 \end{pmatrix} \). Therefore \( \det(A) = -2 - 1/4 = -9/4 < 0 \),
and \( Tr(A) = 1 + 2 = 3 < 0 \). Therefore the eigenvalues of \( A \) are non zero and of opposite signs: Hyperbolic PDE.

(2) \[ u_{xx} - 2u_{xy} + u_{yy} = 0. \] (29)

We have \( A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). Therefore \( \det(A) = 0 \). Therefore at least one of the eigenvalues of \( A \) is zero: Degenerate PDE.

(3) \[ u_{xx} + 2u_{xy} + 2u_{yy} = 0. \] (30)

We have \( A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \). Therefore \( \det(A) = 2 - 1 = 1 > 0 \) and \( Tr(A) = 3 > 0 \). Therefore the eigenvalues of \( A \) are non zero and of the same sign: Elliptic PDE.

Problem 5.

(1) Let’s find the general \( C^2 \) solution of the PDE

\[ u_{xx} - u_{xt} - 6u_{tt} = 0, \] (31)

by reducing it to a system of first order PDEs (by the way, this is an elliptic PDE).

We are looking for \( a, b, c, d \) that formally verify:

\[
\partial_{xx} - \partial_{xt} - 6\partial_{tt} = (a\partial_x + b\partial_t)(c\partial_x + d\partial_t) = ac\partial_{xx} + (ad + bc)\partial_{xt} + bd\partial_{tt}.
\] (32)

So we get the (under-determined) system:

\[
\begin{cases}
ac = 1 \\
ad + bc = -1 \\
bd = -6.
\end{cases}
\] (33)

From the first equation, let us simply take \( a = c = 1 \). Then the system reduces to

\[
\begin{cases}
d + b = -1 \\
bd = -6,
\end{cases}
\] (34)
which gives $b = 2$ and $d = -3$.

Therefore we can write $u_{xx} - u_{xt} - 6u_{tt} = 0$ as $(\partial_x + 2\partial_t)(\partial_x - 3\partial_t)u = 0$. Now let $v = (\partial_x - 3\partial_t)u$. Then $v$ verifies $v_x + 2v_t = 0$ (first order linear PDE!). And we know that the solution writes $v(x,t) = h(t - 2x)$ for some $h \in C^1$.

Now for $u$ we have the system:

$$u_x - 3u_t = h(t - 2x).$$  \hspace{1cm} (35)

Using the method of characteristics, we get the following equations:

$$\begin{cases}
x_r'(s) = 1, & x_r(0) = 0, \\
t_r'(s) = -3, & t_r(0) = r, \\
v_r'(s) = h(t_r(s) - 2x_r(s)), & v_r(0) = \phi(r),
\end{cases}$$  \hspace{1cm} (36)

for some function $\phi \in C^2$.

Therefore we have $x_r(s) = s$, $t_r(s) = -3s + r$, and

$$v_r'(s) = h(-3s + r - 2s) = h(-5s + r),$$  \hspace{1cm} (37)

and by integrating from $s = 0$, we get:

$$v_r(s) = \int_0^s h(-5s' + r)ds' + \phi(r)$$

$$= -\frac{1}{5} \int_r^{5s+r} h(y)dy + \phi(r),$$  \hspace{1cm} (38)

after a change of variables. Now, since we have $s = x$ and $r = t + 3x$, we finally get:

$$u(x,t) = \frac{1}{5} \int_{t-3x}^{t+3x} h(y)dy + \phi(t + 3x)$$

$$= f(t + 3x) + g(t - 2x),$$  \hspace{1cm} (39)

for some $f, g \in C^2$.

Reciprocally, we verify that $u$ of the form $u(x,t) = f(t + 3x) + g(t - 2x)$ for $f, g \in C^2$ indeed solves the PDE.

\textbf{(2)} For an arbitrary $\phi \in C^\infty_c(\mathbb{R}^2)$ we have to show that

$$u(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) + w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = 0.$$  \hspace{1cm} (40)

But from \textbf{Problem 2} of Pset 2, we have:

$$v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v((\partial_x - 3\partial_t)(\phi_x + 2\phi_t)) = 0,$$  \hspace{1cm} (41)

and similarly,

$$w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = w((\partial_x + 2\partial_t)(\phi_x - 3\phi_t)) = 0.$$  \hspace{1cm} (42)