MATH 173: Problem Set 4
Solutions

Problem 1.

(i) For any $\epsilon > 0$, we can find $N > 0$, so that if $j > N$, if $z \in \delta_j \text{supp}\phi$, $|f(x + z) - f(x)| \leq \epsilon$.

\[ |\int f_{y,j}(x)\psi(x)dx - \psi(y)| = |\int \delta_j^{-n}(y/\delta_j)(\psi(x + y) - \psi(x))dy| \leq \epsilon \int \delta_j^{-n}(y/\delta_j)dy = \epsilon, \]

So we have the convergence result.

(ii) Notice:

\[ \partial_m \psi_j = \int \delta_j^{-n}(-1/\delta_j)^m\partial_m\phi((x - y)/\delta_j)\psi(x)dx, \]

for arbitrary sub-index $m$. So $\psi_j$ is $C^\infty$. For the support, since $\psi$ is compactly supported, say, $\text{supp}\psi \subset [-K, K]$, then for $|y| \geq K + \delta_j$, (notice the center has greater distance than the sum of the radius of the supports) the integral is 0. So $\psi_j$ is also compactly supported.

(iii) Recall the calculation:

\[ |\int f_{y,j}(x)\psi(x)dx - \psi(y)| = |\int \delta_j^{-n}(y/\delta_j)(\psi(x + y) - \psi(x))dy| \leq \epsilon \int \delta_j^{-n}(y/\delta_j)|\psi(x + y) - \psi(x)|dy \]

For any $\epsilon > 0$, we have $\delta > 0$ such that, if $|y| < \delta$, $|\psi(x + y) - \psi(x)| \leq \epsilon$, uniformly in $x$. (because of the uniform continuity from $\psi$) then we have $N$ such that $j > N$, then $\delta_j < \delta$, then the integral is less then $\epsilon$, uniformly in $x$. We proved the convergence.

Problem 2. We consider the conservation law:

\[ u_t + (f(u))_x = 0, \quad u(x, 0) = \phi(x), \]

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with \( f \in C^2(\mathbb{R}) \).

Since \( u \) is continuous and \( f \) is \( C^2 \), \( v = f'(u) \) is also continuous. Since \( u \) is \( C^1 \) apart from jump discontinuities in its first derivatives, away from the jumps, \( u_t \) (resp. \( u_x \)) is perfectly defined and continuous. Therefore, since \( f' \) is \( C^1 \), and away from the discontinuities, \( v_t = f''(u)u_t \) (resp. \( v_x = f''(u)u_x \)) is also continuous, \( v \) is \( C^1 \) apart from jump discontinuities in its first derivatives (the same ones as \( u \)). Therefore \( v \) has the same properties as \( u \). Moreover, we have, away from discontinuities:

\[
v_t + vv_x = f''(u)u_t + f'(u)f''(u)u_x = f''(u)(u_t + f'(u)u_x) = f''(u)(u_t + (f(u))_x) = 0. \tag{5}
\]

So \( v \) verifies the Burger’s equation (the Rankine-Hugoniot condition is vacuous: there are no shock since \( v \) is continuous).

If \( f'' > 0 \), \( f \) is strictly convex and \( f' \) is strictly increasing and therefore the inverse function \((f')^{-1}\) exists. We can therefore first solve for \( v \) from the Burger’s equation:

\[
v_t + vv_x = 0, \quad v(x, 0) = f'(\phi(x)), \tag{6}
\]

and then \( u = (f')^{-1}(v) \) is solution of the original PDE.

Suppose now that \( u \) has a jump discontinuity. Then, according to the Rankine-Hugoniot condition:

\[
\xi'(t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{7}
\]

If \( v \) could have been defined as previously, \( v \) would have the same discontinuity \((v = f'(u))\) and again by Rankine-Hugoniot:

\[
\xi'(t) = \frac{\frac{v^2}{2} - \frac{v^2}{2}}{v_+ - v_-} = \frac{1}{2}(v_+ + v_-) = \frac{1}{2}(f'(u_+) + f'(u_-)). \tag{8}
\]

But in general,

\[
\frac{f(u_+) - f(u_-)}{u_+ - u_-} \neq \frac{1}{2}(f'(u_+) + f'(u_-)) \tag{9}
\]

(consider for instance \( f(x) = e^x \)) and the statement is FALSE.

**Problem 3.** Let \( f(x, t) = u^2 + c(x)^2 |\nabla u|^2 + q(x)u^2 \), so that we can simply write \( E(t) = \int_{|x-x_0| < R_0 - ct} f(x, t) dx \).
To investigate the variation of $E(t)$, we would like to derive $E$ with respect to $t$. The difficulty here is that $t$ appears in the very domain of integration, so that we have an integral with parameter. As the hint suggests, let us first look at the 1D case. Then

$$E(t) = \int_{|x-x_0|<R_0-c_2t} f(x,t)dx,$$

and at this point, we can use the following result (exercise):

**Lemma 0.1.** Suppose $a(t)$ and $b(t)$ are $C^1$ functions, and $g(x,t)$ continuous in $x$ and $C^1$ in $t$. Then if $G(t) = \int_{a(t)}^{b(t)} g(x,t)dx$, then

$$G'(t) = b'(t)f(b(t),t) - a'(t)f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial g}{\partial t}(x,t)dx.$$

Now by applying it in our problem, we get:

$$E'(t) = -c_2f(x_0 - R_0 + c_2t,t) - c_2f(x_0 + R_0 - c_2t,t) + \int_{x_0-R_0+c_2t}^{x_0+R_0-c_2t} \frac{\partial f}{\partial t}(x,t)dx.$$

Therefore we see that when the moving domain is in 1D, it is rather simple to manage the derivative of $E(t)$. Now let us go back to the general case. If we look at the moving domain in our case, we have that $\{|x-x_0|<R_0-c_2t\} = B(x_0, R_0 - c_2t)$, the ball centered at $x_0$ and of radius $R_0 - c_2t$. Therefore we have radial symmetry which naturally inclines us to use the polar change of coordinates to get:

$$E(t) = \int_{|x-x_0|<R_0-c_2t} f(x,t)dx,$$

where $S^{n-1} = \{x \in \mathbb{R}^n, |x| = 1\}$ is the unit sphere.

Then applying the lemma on $E(t)$, we get:

$$E'(t) = -c_2\int_{S^{n-1}} f(x_0 + (R_0-c_2t)\omega,t)(R_0-c_2t)^{n-1}d\omega$$

$$+ \int_{r=R_0-c_2t}^{r=R_0+c_2t} \left( \int_{S^{n-1}} \frac{\partial f}{\partial t}(x_0 + r\omega,t)r^{n-1}d\omega \right)dr$$

$$= -c_2\int_{|x-x_0|=R_0-c_2t} f(x,t)dx + \int_{|x-x_0|<R_0-c_2t} \frac{\partial f}{\partial t}(x,t)dx,$$
where we used again change of variables/coordinates.

Now we can write, replacing \( f(x,t) \) by its value:

\[
E'(t) = -c_2 \int_{|y-x_0|=R_0 - c_2 t} \left( u_t^2 + c(y)^2 |\nabla u|^2 + q(y)u^2 \right) dy + 2 \int_{|x-x_0|<R_0 - c_2 t} (u_t u_{tt} + c(x)^2 \nabla u \cdot \nabla u_t + q(x)u_t) dx,
\]

(15)

And here we see that from \( c(x)^2 \nabla u \cdot \nabla u_t \), we would like to get back \( u_t \) and \( \nabla \cdot (c(x)^2 \nabla u) \), which invites us to use the divergence theorem to get:

\[
\int_{|x-x_0|<R_0 - c_2 t} c(x)^2 \nabla u \cdot \nabla u_t dx = \int_{|y-x_0|=R_0 - c_2 t} c(y)^2 \frac{\partial u}{\partial \nu} u_t dy - \int_{|x-x_0|<R_0 - c_2 t} \nabla \cdot (c(x)^2 \nabla u) u_t dx,
\]

(16)

so that:

\[
E'(t) = \int_{|y-x_0|=R_0 - c_2 t} \left( 2c(y)^2 \frac{\partial u}{\partial \nu} u_t - c_2 \left( u_t^2 + c(y)^2 |\nabla u|^2 + q(y)u^2 \right) \right) dy + 2 \int_{|x-x_0|<R_0 - c_2 t} u_t (u_{tt} - \nabla \cdot (c(x)^2 \nabla u) + q(x)u) dx
\]

\[
= \int_{|y-x_0|=R_0 - c_2 t} \left( 2c(y)^2 \frac{\partial u}{\partial \nu} u_t - c_2 \left( u_t^2 + c(y)^2 |\nabla u|^2 + q(y)u^2 \right) \right) dy,
\]

(17)

since \( u \) verifies the PDE. Now, using the same "trick" as in class, we have that:

\[
\int_{|y-x_0|=R_0 - c_2 t} 2c(y)^2 \frac{\partial u}{\partial \nu} u_t dy \leq \int_{|y-x_0|=R_0 - c_2 t} c(y)2c(y) \left| \frac{\partial u}{\partial \nu} \right| |u_t| dy \\
\leq c_2 \int_{|y-x_0|=R_0 - c_2 t} 2c(y) |\nabla u| |u_t| dy \\
\leq c_2 \int_{|y-x_0|=R_0 - c_2 t} \left( c(y)^2 |\nabla u|^2 + u_t^2 \right) dy.
\]

(18)

because \( c(x) \geq 0 \) and \( c(x) < c_2 \). And finally, putting everything together, we get that:

\[
E'(t) \leq -c_2 \int_{|y-x_0|=R_0 - c_2 t} q(y)u^2 dy
\]

\[
\leq 0,
\]

(19)

and \( E \) is indeed non-increasing with \( t \).
(ii) Now let’s suppose that suppφ, suppψ ⊂ \{|x| ≤ R\} = B(0, R). Let us consider \((x_0, t_0)\) such that \(t_0 \geq 0\) and \(x_0 > R + c_2 t_0\). The aim here is to prove that \(u(x_0, t_0) = 0\).

To understand what is going on let us first draw a picture:

So if you take \(R_0 = c_2 t_0\) and consider, for \(t < \frac{R_0}{c_2} = t_0\)

\[
E(t) = \int_{|x-x_0|<c_2(t_0-t)} \left( u_t^2 + c(x)^2 |\nabla u|^2 + q(x) u^2 \right) dx, \tag{20}
\]

then from (i), \(E\) is non-increasing. Furthermore, since suppφ, suppψ ⊂ \{|x| ≤ R\} while \{|x-x_0| < c_2 t_0\} ⊂ \{|x| > R\}, we can assert that

\[
E(0) = \int_{|x-x_0|<c_2 t_0} \left( \psi^2 + c(x)^2 |\nabla \phi|^2 + q(x) \phi^2 \right) dx = 0. \tag{21}
\]

Moreover, from the expression of \(E(t)\), \(E\) is non-negative, and therefore \(E(t) = 0\), \(0 ≤ t < t_0\). Consequently, \(u_t = 0\), \(\nabla u = 0\) as \(c(x) ≥ c_1 > 0\) and \(q(x) ≥ 0\). This implies that \(u\) is constant in the cone \(\{|x-x_0| ≤ c_2(t_0-t)\}\). Finally, noticing that \(u(x_0, 0) = φ(x_0) = 0\) since \(x_0 > R\), we conclude that the constant is zero and \(u(x_0, t_0) = 0\). Therefore the wave indeed propagates at finite speed \(≤ c_2\).

(iii) Let’s suppose that we have \(u_1\) and \(u_2\) solutions of the PDE. Then by linearity, \(w\) defined as \(w = u_2 - u_1\) solves the following PDE:

\[
\begin{cases}
  w_t - \nabla \cdot (c(x)^2 \nabla w) + q(x) w = 0, \\
  w(x, 0) = 0, \\
  w_t(x, 0) = 0.
\end{cases} \tag{22}
\]

Hence, from part (ii), we directly get that for any \((x_0, t_0)\), \(w(x_0, t_0) = 0\). Therefore \(u_1\) and \(u_2\) are identical and there is at most one real-value \(C^2\) solution of the PDE.

**Problem 4.** Consider the wave equation on \(\mathbb{R}^n\):

\[
\begin{cases}
  u_{tt} - c^2 \Delta u = f, \\
  u(x, 0) = \phi(x), \\
  u_t(x, 0) = \psi(x),
\end{cases} \tag{23}
\]

and we write \(x = (x', x_n)\) with \(x' = (x_1, x_2, ..., x_{n-1})\).

(i) We suppose here that \(f, \phi\) and \(\psi\) are all even functions of \(x_n\). Following the hint, let’s consider \(v(x', x_n, t) = u(x', x_n, t) - u(x', -x_n, t)\) and let’s find a PDE for \(v\).

Using the assumption on the data, we get:

\[
\bullet \ v_{tt}(x', x_n, t) = u_{tt}(x', x_n, t) - u_{tt}(x', -x_n, t).
\]
\[ \Delta v(x', x_n, t) = \Delta u(x', x_n, t) - \Delta u(x', -x_n, t), \]
\[ f(x', x_n, t) - f(x', -x_n, t) = 0, \]
\[ v(x', x_n, 0) = \phi(x', x_n) - \phi(x', -x_n) = 0, \]
\[ v_t(x', x_n, 0) = \psi(x', x_n) - \psi(x', -x_n) = 0. \]

Therefore \( v \) verifies the following PDE:

\[
\begin{align*}
\begin{cases}
v_{tt} - c^2 \Delta v = 0, \\
v(x, 0) = 0, \\
v_t(x, 0) = 0,
\end{cases}
\end{align*}
\]

(24)

and from **Problem 4**, this implies that \( v \) is identically zero, which gives

\[ u(x', x_n, t) = u(x', -x_n, t), \]

(25)

meaning the solution \( u \) conserves the symmetry of the data.

(ii) We suppose here that \( f, \phi \) and \( \psi \) are all odd functions of \( x_n \). Let’s consider \( w(x', x_n, t) = u(x', x_n, t) + u(x', -x_n, t) \) and let’s find a PDE for \( w \).

Using the assumption on the data, we get:

\[ w_{tt}(x', x_n, t) = u_{tt}(x', x_n, t) + u_{tt}(x', -x_n, t), \]
\[ \Delta w(x', x_n, t) = \Delta u(x', x_n, t) + \Delta u(x', -x_n, t), \]
\[ f(x', x_n, t) + f(x', -x_n, t) = 0, \]
\[ v(x', x_n, 0) = \phi(x', x_n) + \phi(x', -x_n) = 0, \]
\[ v_t(x', x_n, 0) = \psi(x', x_n) + \psi(x', -x_n) = 0. \]

Therefore \( w \) verifies the following PDE:

\[
\begin{align*}
\begin{cases}
w_{tt} - c^2 \Delta w = 0, \\
w(x, 0) = 0, \\
w_t(x, 0) = 0,
\end{cases}
\end{align*}
\]

(26)

and from **Problem 4**, this implies that \( w \) is identically zero, which gives

\[ u(x', x_n, t) = -u(x', -x_n, t), \]

(27)

meaning the solution \( u \) conserves the symmetry of the data.

(iii) From (ii), we have that \( u(x', 0, t) = -u(x', -0, t) \) and by continuity of \( u \), \( u(x', 0, t) = 0. \)
From (i), we have that \( u(x', x_n, t) = u(x', -x_n, t) \) and since \( u \) is \( C^1 \), we get that \( \partial_{x_n} u(x', x_n, t) = -\partial_{x_n} u(x', -x_n, t) \). Now taking \( x_n = 0 \), we conclude that \( \partial_{x_n} u(x', 0, t) = -\partial_{x_n} u(x', 0, t) \) and \( \partial_{x_n} u(x', t) = 0 \).

**Problem 5.** We consider \( u \in C^2(\mathbb{R}^n) \), such that \( \Delta u = 0 \) and \( \sup_{|x|>R} |u(x)| \to 0 \) as \( R \to +\infty \) (uniformly vanishing at infinity).

For any \( R > 0 \), we can apply the maximum principle on the ball \( B(0, R) \) to get:

\[
\sup_{|x|<R} u(x) = \sup_{|x|=R} u(x). \tag{28}
\]

Now let's fix \( \epsilon > 0 \). Since \( u \) is uniformly vanishing at infinity, there exists \( R_0 > 0 \) such that for all \( R > R_0 \), \( |u(x)| < \epsilon \) if \( |x| = R \) (in particular, \( u(x) \leq |u(x)| < \epsilon \)). Hence

\[
\sup_{|x| \in \mathbb{R}^n} u(x) \leq \lim_{R \to +\infty} \sup_{|x| \leq R} u(x) = \lim_{R \to +\infty} \sup_{|x|=R} u(x) < \epsilon. \tag{29}
\]

Since this is true for any \( \epsilon > 0 \) and for any \( x \in \mathbb{R}^n \), we conclude that

\[
u(x) \leq \sup_{|x| \in \mathbb{R}^n} u(x) \leq 0. \tag{30}\]

Now we consider \(-u(x)\). It is as well uniformly vanishing at infinity and verifies the maximum principle. Therefore, by the same arguments, we get that

\[
-u(x) \leq \sup_{|x| \in \mathbb{R}^n} (-u(x)) \leq 0 \Leftrightarrow u(x) \geq 0. \tag{31}\]

Therefore we conclude that \( u(x) = 0 \).

Suppose \( u \) and \( v \) are \( C^2(\mathbb{R}^n) \) and uniformly vanishing at infinity. Suppose furthermore that \( \Delta u = f, \Delta v = f \) with \( f \) given. Consider \( w = u - v \). Then \( w \) has the same properties as \( u \) and \( v \), and satisfies \( \Delta w = 0 \). But from the first part of the problem, this implies that \( w = 0 \), which means \( u = v \). Therefore the solution of the Laplace equation on \( \mathbb{R}^n \) for that class of functions is unique (if it exists).

**Problem 6.** Multiply the equation by \( u \) and via integration by parts:

\[
\int \nabla u^T A \nabla u + \int q u^2 = \int f u. \tag{32}
\]

Notice \( q \geq 0 \) and by ellipticity, and by AGM: we have:

\[
\int c_0 |\nabla u|^2 \leq C \int f^2 + \frac{1}{C} \int u^2. \tag{33}\]

Now apply Poincaré's Inequality:

\[
\frac{2}{C} \int |\nabla u|^2 + \frac{c_0}{2} \int u^2 \leq C \int f^2 + \frac{1}{C} \int u^2. \tag{34}\]
Then subtract the $u^2$ on the right, and rearranging the coefficients, we get the equation we want.