**Problem 1.** Let \( f \in L^1(\mathbb{R}^n) \) and \( a \in \mathbb{R}^n \). The whole problem is a matter of change of variables with integrals.

(i) 
\[
(\hat{f}_a)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x-a) \, dx = \int_{\mathbb{R}^n} e^{-i(a+y) \cdot \xi} f(y) \, dy = e^{-i\alpha \cdot \xi} (\hat{f}_a)(\xi).
\]

(ii) 
\[
(\hat{g}_a)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} g_a(x) \, dx = \int_{\mathbb{R}^n} e^{-i(\xi-a) \cdot x} f(x) \, dx = (\hat{f})(\xi-a).
\]

(iii) 
\[
(\hat{f}^{-1}_a)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(x) \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x+\alpha+y) \cdot \xi} f(y) \, dy = e^{i\alpha \cdot \xi} (\hat{f}^{-1}_a)(\xi).
\]

(iv) 
\[
(\hat{g}^{-1}_a)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} g_a(x) \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(\xi+\alpha) \cdot x} f(x) \, dx = (\hat{f}^{-1})(\xi + a).
\]
Problem 2. Let $f \in \mathcal{C}^1(\mathbb{R}^n)$, and $|x|^N f(x)$, $|x|^N \partial_j f(x)$ bounded with $N > n$.

Let’s write $x = (x_1, \ldots, x_j, \ldots, x_n)$ and take $h_j = (0, \ldots, h_j, \ldots, 0)$ = he$_j$ (zero everywhere except for the $j^{th}$ coordinate).

We have

$$ (\mathcal{F}(\partial_j f)) (\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} (\partial_j f)(x) dx = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \lim_{h \to 0} \frac{f(x + h e_j) - f(x)}{h} dx $$

If we can switch limit and integral, we are done. Therefore we need to justify the inversion.

Let’s take $h_j^k = (0, \ldots, h_j^k, \ldots, 0) = h^k e_j$, with $(h^k)_{k \geq 0}$ such that $h^k \to 0$ as $k \to +\infty$. Now we can consider $g_k(x) = \frac{f(x + h_j^k) - f(x)}{h^k}$.

- $g_k$ is integrable for all $k \geq 0$ since $f$ continuous and $|x|^N f(x)$ is bounded,
- $g_k(x) \to \partial_j f(x)$ as $k \to +\infty$ pointwise,
- for all $k \geq 0$, $|g_k(x)| \leq \phi(x)$, with $\phi \in L^1(\mathbb{R}^n)$, $\phi(x) \geq 0$. Indeed, by the mean value theorem, there exists $\eta^k_j \in (x^k_j, x_j^k + h^k)$ such that $g_k(x) = \partial_j f(x_1, \ldots, \eta^k_j, \ldots, x_n)$. Now, for instance, for $|x| < 1$, there exists $M_1 > 0$ such that $|\partial_j f(x)| \leq M_1$ ($\partial_j f$ continuous), and for $|x| > 1$, there exists $M_2 > 0$ such that $|\partial_j f(x)| \leq \frac{M_2}{|x|^N}$ by assumption.

Hence, $|g_k(x)| \leq M_1 \chi_{\{|x|<1\}} + \frac{M_1}{|x|^N} \chi_{\{|x|>1\}} = \phi(x)$, with $\chi$ the indicator function.

Therefore, we can use the Lebesgue Dominated Convergence Theorem (LDCT) on $h_k(x; \xi) = e^{-ix \cdot \xi} g_k(x)$ (same conclusions since $|e^{-ix \cdot \xi}| = 1$) and write:

$$ (\mathcal{F}(\partial_j f)) (\xi) = \int_{\mathbb{R}^n} \lim_{k \to +\infty} h_k(x; \xi) dx $$

$$ = \lim_{k \to +\infty} \int_{\mathbb{R}^n} h_k(x; \xi) dx $$

$$ = \lim_{k \to +\infty} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \frac{f(x + h^k e_j) - f(x)}{h^k} dx $$

$$ = \lim_{k \to +\infty} \frac{e^{ih^k \cdot \xi}}{h^k} (\mathcal{F} f)(\xi) - (\mathcal{F} f)(\xi) $$

$$ = \left( \lim_{k \to +\infty} \frac{e^{ih^k \cdot \xi} - 1}{h^k} \right) \mathcal{F} f(\xi) $$

$$ = \left( \lim_{k \to +\infty} \frac{e^{ih^k \cdot \xi} - 1}{h^k} \right) \mathcal{F} f(\xi) $$

$$ = i \xi_j (\mathcal{F} f)(\xi) $$
In short, if \( f \) is \( C^1 \) and \( |x|^N f, |x|^N \partial_j f \) are bounded, with \( N > n \), we can invert integral and limit. Also, we could have used integration by parts to get to the result.

An alternative of the LDCT is to use the following argumentation.

\[
(1 + |x|)^{N'} \frac{f(x + h_j) - f(x)}{h} \quad \text{converges uniformly on the real line corresponding to the } x_j\text{-axis to } (1 + |x|)^{N'} \partial_j f(x) \text{ for } N' < N.
\]

Indeed, the difference quotient is \( \int_0^1 \partial_j f(x + sh_j)ds \) by Taylor’s theorem (or simply the fundamental theorem of calculus with a change of variable), and by assumption this integral is bounded by \( C(1 + |x|)^{-N} \). So first \( (1 + |x|^N) \frac{f(x + h_j) - f(x)}{h} \) is uniformly bounded (in \( |h| \leq 1, h \neq 0 \) and \( x \in \mathbb{R}^n \)).

Further, it differs from \( \partial_j f(x) \) by \( \int_0^1 (\partial_j f(x + sh) - \partial_j f(x))ds \). Since \( \partial_j f \) is continuous, it is uniformly continuous on compact sets, and since \( (1 + |x|)^{N'} \partial_j f \) is bounded, \( (1 + |x|)^{N'} \partial_j f \) is uniformly continuous for \( N' < N \). (The extra decay factor \( (1 + |x|)^{(N' - N)} \) makes sure it goes to 0 at infinity, which is why uniform continuity is easy.) Thus, given \( \epsilon > 0 \), for sufficiently small \( h \), and for \( s \in [0, 1] \), one has for all \( x \in \mathbb{R}^n \),

\[
(1 + |x|)^{N'} |\partial_j f(x + sh_j) - \partial_j f(x)| < \epsilon
\]

Thus,

\[
\left| (1 + |x|)^{N'} \frac{f(x + h_j) - f(x)}{h} - (1 + |x|)^{N'} \partial_j f(x) \right| < \epsilon
\]

for sufficiently small \( h \), and so, with \( (1 + |x|)^{-N'} \) being integrable,

\[
\int \left| \frac{f(x + h_j) - f(x)}{h} - \partial_j f(x) \right| dx < C'\epsilon,
\]

and thus \( \left| \int \frac{f(x + h_j) - f(x)}{h}dx - \int \partial_j f(x)dx \right| < C'\epsilon \) for small \( h \). This gives the desired convergence (since you can add the extra multiplicative coefficient \( e^{-ix\xi} \) with no significant effect on the conclusions).

Problem 3.

(i) \( H(a - |x|) = 1 \) if \(-a < x < a\), and 0 otherwise. Therefore

\[
(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} H(a - |x|)dx = \int_{-a}^{a} e^{-ix\xi}dx = \frac{e^{i\alpha\xi} - e^{-i\alpha\xi}}{i\xi}.
\]
(ii) $H(x)e^{-ax}$ is 0 if $x < 0$. Therefore:

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi}e^{-ax}H(x)dx = \int_{0}^{+\infty} e^{-(i\xi+a)x}dx = \frac{1}{i\xi+a}. \quad (10)$$

(iii)

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix|\xi|^n e^{-a|x|}}dx = \int_{0}^{+\infty} x^n e^{-(i\xi+a)x}dx + \int_{-\infty}^{0} e^{-(i\xi-a)x}dx$$

If we consider the first integral, doing integrations by part $n$ times, we get:

$$\int_{0}^{+\infty} x^n e^{-(i\xi+a)x}dx = (-1)^n \frac{n!}{(-a-i\xi)^n} \int_{0}^{+\infty} e^{-(i\xi-a)x}dx$$

$$= (-1)^{n+1} \frac{n!}{(-a-i\xi)^{n+1}} = \frac{n!}{(a+i\xi)^{n+1}} \quad (12)$$

Similarly, for the second term, we have

$$\int_{-\infty}^{0} (-x)^n e^{-(i\xi-a)x}dx = \frac{n!}{(a-i\xi)^{n+1}} \quad (13)$$

Therefore

$$(\mathcal{F}f)(\xi) = \frac{n!}{(a+i\xi)^{n+1}} + \frac{n!}{(a-i\xi)^{n+1}} \quad (14)$$

(iv) We can write

$$\frac{1}{1+x^2} = \frac{1}{2} \left( \frac{1}{1+ix} + \frac{1}{1-ix} \right). \quad (15)$$

Furthermore, we have

$$\left(\mathcal{F}^{-1}e^{-|x|}\right)(\xi) = \frac{1}{2\pi} \left( \int_{-\infty}^{0} e^{ix\xi}e^x dx + \int_{0}^{+\infty} e^{ix\xi}e^{-x} dx \right)$$

$$= \frac{1}{2\pi} \left( - \int_{-\infty}^{0} e^{-iy^2}e^{-y} dy - \int_{0}^{+\infty} e^{-iy^2}e^y dy \right)$$

$$= \frac{1}{2\pi} \left( \int_{0}^{+\infty} e^{-iy^2}e^y dy + \int_{-\infty}^{0} e^{-iy^2}e^{-y} dy \right)$$

$$= \frac{1}{2\pi} \left( \frac{1}{1+i\xi} + \frac{1}{1-i\xi} \right). \quad (16)$$
by change of variable and using the result from part (iii) with \( n = 0 \).

Now using the Fourier inversion formula, we deduce that:

\[
\left( \mathcal{F} \frac{1}{1 + x^2} \right) (\xi) = \left( \mathcal{F} \frac{1}{2} \left( \frac{1}{1 + ix} + \frac{1}{1 - ix} \right) \right) (\xi) \\
= \pi \left( \mathcal{F} \mathcal{F}^{-1} e^{-|x|} \right) (\xi) \\
= \pi e^{-|\xi|}.
\] (17)

Problem 4. We have that

\[
(\mathcal{F} f)(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} |x|^n e^{-a|x|} dx.
\] (18)

Let \( T \) be a linear transformation such that we rotate the \( z \)-axis to \( \xi \) (we fix \( \xi \in \mathbb{R}^3 \)). Therefore \( T([0, 0, 1]) = \xi \). Let \( T([1, 0, 0]) = \xi_1 \) and \( T([0, 1, 0]) = \xi_2 \).

We can use spherical coordinates such that \( x = (x_1, x_2, x_3) \) into \( (r, \theta, \phi) \), where \( r = |x| \), \( \theta \) is the angle between \( \xi \) and \( x \), and \( \phi \) is an angle of rotation about \( \xi \) from \( \xi_2 \).

Since \( x \cdot \xi = |x||\xi| \cos(\theta) \), we obtain that

\[
(\mathcal{F} f)(\xi) = \int_{\theta=0}^{\pi} \int_{r=0}^{+\infty} \int_{\phi=0}^{2\pi} e^{-ir|\xi| \cos(\theta)} r^n e^{-ar r^2 \sin(\theta)} r^2 \sin(\theta) dr d\theta d\phi
\] (19)

Problem 5.

(i) By integration by part, we get

\[
(\mathcal{F}_z D_z f)(y, \zeta) = \frac{1}{i} \int_{\mathbb{R}^k} e^{-iz \cdot \zeta} \partial_z f(y, z) dz \\
= -\frac{1}{i} \int_{\mathbb{R}^k} \partial_z \left( e^{-iz \cdot \zeta} \right) f(y, z) dz \\
= \zeta_j \int_{\mathbb{R}^k} e^{-iz \cdot \zeta} f(y, z) dz \\
= \zeta_j (\mathcal{F}_z f)(y, \zeta)
\] (20)
(ii) By definition, we have that
\[
(F_z D_{y_j} f)(y, \zeta) = \frac{1}{i} \int_{R^k} e^{-iz \cdot \zeta} \partial_{y_j} f(y, z) dz \tag{21}
\]

Now since \( f \) is \( C^1 \), and \( |z|^K f, |z|^K \partial_{x_j} f \) are bounded with \( K > k \), using the same argumentation as in Pb2, we can switch the derivative and the integral sign to finally obtain
\[
(F_z D_{y_j} f)(y, \zeta) = D_{y_j} \int_{R^k} e^{-iz \cdot \zeta} f(y, z) dz = (D_{y_j} F_z f)(y, \zeta). \tag{22}
\]

**Problem 6.** I will give the solution in the next homework problem 1.