For this homework, we explore more advanced limits, as well as vertical and horizontal asymptotes.

**Problem 1**: For each item, list all of the functions from I-IV below that meet the description. There may be more than one function for each description, or none at all.

(a) Horizontal asymptote of \( y = 1 \).

(b) The \( x \)-axis is a horizontal asymptote.

(c) Vertical asymptote at \( x = 1 \) and \( x = -1 \).

(d) Horizontal asymptote of \( y = -1 \).

(e) The limit as \( x \to 1 \) exists.

I. \( y = \frac{x - 1}{x^2 + 1} \)  
II. \( y = \frac{-x^2 + 1}{x^2 + 1} \)  
III. \( y = \frac{x^2 + 1}{x^2 - 1} \)  
IV. \( y = \frac{x + 1}{x^2 - 1} \)

**Problem 2**: Consider the function

\[
 f(x) = \begin{cases} 
 e^x & 0 \leq x \leq 1 \\
 4 + (x - 1)^2 & 1 < x \leq 2 
\end{cases}
\]

Can we use the Intermediate Value Theorem to show that there is a number \( c \) with \( 0 \leq c \leq 2 \) so that \( f(c) = 3 \)? If yes, write out the complete argument. If not, explain why.

**Problem 3**: Find \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) for \( f(x) = \frac{3e^x + 2}{2e^x + 3} \).

**Problem 4**: Find all values of \( k \) so that the limit \( \lim_{x \to \infty} \frac{x^3 - 6}{x^k + 3} \) exists.

**Problem 5**: Find all horizontal and vertical asymptotes for the rational function \( f(x) = \frac{5x - 2}{2x + 3} \). Note: for the vertical asymptotes \( x = a \), you should calculate \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \).

**Problem 6**: If possible, find \( k \) so that the following function is continuous for all real numbers:

\[
 f(x) = \begin{cases} 
 k \cos x & x \leq 0 \\
 e^x - k & x > 0 
\end{cases}
\]

[More on next page]
For problems 7-12, determine the requested limit or state that it does not exist. If the limit does not exist, describe the behavior of the function near that point (for example, show that \( \lim_{x \to a} f(x) = \pm \infty \))

**Problem 7:**

\( (a) \lim_{x \to 2} \frac{x - 4}{x - 2} \quad (b) \lim_{x \to 3^+} \frac{x - 3}{x + 3} \)

**Problem 8:** \( \lim_{x \to -\infty} \frac{\sin^4(x^3)}{x^2} \). Note: the notation \( \sin^4(\theta) \) is a shorthand for \( (\sin \theta)^4 \).

**Problem 9:**

\( (a) \lim_{x \to 0^-} \left( \frac{1}{x} - \frac{1}{|x|} \right) \quad (b) \lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{|x|} \right) \)

**Problem 10:** \( \lim_{x \to \infty} (\ln(2 + x) - \ln(1 + 2x)) \)

*Hint: log rules will be helpful here.*

**Problem 11:**

\( (a) \lim_{x \to \infty} 4x \left( 3x - \sqrt{9x^2 + 1} \right) \quad (b) \lim_{x \to -\infty} 4x \left( 3x - \sqrt{9x^2 + 1} \right) \)

**Problem 12:** \( \lim_{x \to 1} h(x) \) where \( h(x) = \begin{cases} \sqrt{1 - x^2} & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ 2 & x = 2 \end{cases} \)