This homework assignment explores modeling with differential equations. Relevant textbook sections are 11.5, 11.6, and 11.7.

*Note: When we ask you to “write a differential equation for $[a function y = f(t)]$,” we are looking for something like $\frac{dy}{dt} = [\text{stuff involving } y \text{'s and/or } t \text{'s}].$"

**Problem 1:** A bank account that earns 10% interest compounded continuously has an initial balance of zero. Money is deposited into the account at a constant rate of $1000$ per year.

(a) Write a differential equation that describes the rate of change of the balance $B = f(t)$.

(b) Solve the differential equation to find the balance as a function of time.

**Problem 2:** A spherical snowball melts at a rate proportional to its surface area.

(a) Write a differential equation for its volume, $V$.

*Hint: it may be useful to remember that the volume of a sphere is $\frac{4}{3}\pi r^3$ and the surface area is $4\pi r^2$.

(b) If the initial volume is $N$ (some positive number), solve the differential equation.

(c) Is your solution increasing or decreasing? Concave up or concave down?

(d) When does the snowball disappear?

**Problem 3:** Let $L$, a constant, be the number of people who would like to see a newly released movie, and let $N(t)$ be the number of people who have seen it during the first $t$ days since its release. The rate that people first go see the movie, $\frac{dN}{dt}$ (in people/day), is proportional to the number of people who would like to see it but haven’t yet. Write and solve a differential equation describing $\frac{dN}{dt}$ where $t$ is the number of days since the movie’s release. Your solution will involve $L$ and a constant of proportionality $k$.

**Problem 4:** As you know, when a course ends, students start to forget the material they have learned. One model (called the Ebbinghaus model) assumes that the rate at which a student forgets material is proportional to the difference between the material currently remembered and some positive constant, $a$.

(a) Let $y = f(t)$ be the fraction of the original material remembered $t$ weeks after the course has ended. Set up a differential equation for $y$. Your equation will contain two constants: the constant $a$ (which is less than $y$ for all $t$) and the constant of proportionality $k$. Be sure to specify if $k$ is positive, negative, or either.

(b) Solve the differential equation.

(c) Describe the practical meaning (in terms of the amount remembered) of the constants in the solution $y = f(t)$. 

Problem 5: When people smoke, carbon monoxide is released into the air. In a room of volume 60 m$^3$, air containing 5% carbon monoxide is introduced at a rate of 0.002 m$^3$/min. (This means that 5% of the volume of the incoming air is carbon monoxide.) The carbon monoxide mixes immediately with the rest of the air, and the mixture leaves the room at the same rate as it enters.

(a) Write a differential equation for $Q(t)$, the amount of carbon monoxide at time $t$, in minutes.

(b) Let $c(t)$ be the concentration of carbon monoxide at time $t$, in minutes. Using your answer from part (a), write a differential equation for $c(t)$.

(c) Solve the differential equation for $c(t)$, assuming there is no carbon monoxide in the room initially.

(d) What happens to the value of $c(t)$ in the long run?

Problem 6: A population satisfies

$$\frac{dP}{dt} = 0.025P - 0.00005P^2.$$ 

(a) What are the equilibrium values for $P$?

(b) For each of the following initial conditions, describe the long-term behavior of the solution. Will $P$ increase, decrease, or stay the same as $t$ increases? Will the population increase or decrease without bound or to a limiting value? If to a limiting value, what is it? Note: You can answer this question WITHOUT solving the differential equation.

(i) $P_0 = 100$

(ii) $P_0 = 400$

(iii) $P_0 = 500$

(iv) $P_0 = 800$

Problems 7-10: Find the specific solutions to the following initial value problems.

Problem 7: \[ \frac{dy}{dx} + xy^2 = 0, \quad y(1) = 1. \]

Problem 8: \[ \frac{dr}{dt} = (1 + \ln t)r, \quad r(1) = 1. \]

Problem 9: \[ \frac{df}{dx} = \sqrt{xf}, \quad f(1) = 1. \]

Problem 10: \[ \frac{dy}{dx} = e^{x-y}, \quad y(0) = 1. \]

Problems 11 & 12: Evaluate the following integrals.

Problem 11: \[ \int_1^4 \frac{\ln x}{x^2} \, dx. \]

Problem 12: \[ \int \frac{1}{\sqrt{4w^2 - 8}} \, dw. \]

There is a table of integrals on the next page, if you need them.
Table of potentially useful integrals

Note: If you use one of these formulas, please cite it by number in your solution.

1. \( \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C, \ a \neq 0 \)

2. \( \int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \left( \frac{x}{a} \right) + C, \ a \neq 0 \)

3. \( \int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{a-b} \ln |x-a| - \ln |x-b| + C, \ a \neq b \)

4. \( \int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{a-b} [(ac + d) \ln |x-a| - (bc + d) \ln |x-b|] + C, \ a \neq b \)

5. \( \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C, \ a > 0 \)

6. \( \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left| x + \sqrt{x^2 + a^2} \right| + C, \ a > 0 \)

7. \( \int \sqrt{a^2 + x^2} \, dx = \frac{1}{2} \left( x \sqrt{a^2 + x^2} + a^2 \int \frac{1}{\sqrt{a^2 + x^2}} \, dx \right) + C, \ a > 0 \)

8. \( \int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left( x \sqrt{x^2 - a^2} - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C, \ a > 0 \)