Final Practice Problems

I. Basic Concepts and Techniques of Integration

1. The indefinite integral \( \int \frac{dx}{\sqrt{x}} \) is equal to which of the following?

   a. \( 2\sqrt{x} + C \)
   b. \( \ln(\sqrt{x}) + C \)
   c. Both (a.) and (b.) are correct.
   d. Neither (a.) nor (b.) is correct.

2. Evaluate:
   a. \( \int \frac{e^x + 1}{e^x} \, dx \)
   b. \( \int \frac{e^x}{e^x + 1} \, dx \)

3. Evaluate:
   a. \( \int x \sin(x^2) \, dx \)
   a. \( \int x^2 \sin(x) \, dx \)

4. If \( \int x \cos(x) \, dx = x \sin x + \cos x + C \), then what is \( \int e^x \cos(e^x) \, dx \) equal to?

   a. \( e^x (x \sin x + \cos x) + C \)
   b. \( e^x \sin(e^x) + \cos(e^x) + C \)
   c. \( \sin(e^x) + C \)
   d. None of the above.

5. Which of these integrals cannot be evaluated using basic techniques (i.e. without tables) covered in class? Evaluate those you left uncircled.

   a. \( \int \sqrt{x^3 + 1} \, dx \)
   b. \( \int e^{1/x} \, dx \)
   c. \( \int e^{-x} \, dx \)
d. \[ \int e^{x^2} \, dx \]

e. \[ \int \frac{\sin x}{\cos^4 x} \, dx \]

f. \[ \int \ln \left( \frac{1}{x} \right) \, dx \]

g. \[ \int \frac{dx}{\ln x} \]

h. \[ \int \frac{dx}{x \ln x} \]

i. \[ \int \frac{2x}{x^2 + 4x + 4} \, dx \]

j. \[ \int \sqrt{1 + \sin^2 x} \, dx \]

6. Find \( a, b, c \): 
\[ \int_0^1 x^4 \sqrt{x^5 + 1} \, dx = c \int_a^b \sqrt{u} \, du. \]

II. Applications of Integration

7. Find the area bounded above by \( y = x \cos(x^2) \) and below by \( y = x \sin(x^2) \) as illustrated in the figure.

8. Let \( R \) be the region bounded above by \( y = (x^2 + 1)^{-1/2} - \frac{1}{\sqrt{2}} \), on the left by \( x = -1 \) and on the right by \( x = 1 \). Let \( S \) be the solid whose base is \( R \) and whose cross-sections perpendicular to the \( x \)-axis are squares.
a. Find the area of \( R \). You may use table entries.

b. Find the volume of \( S \).

9. Suppose that \( x^2 y' = y \sin(1/x) \) with \( y(2/\pi) = 1 \). Solve for \( y \). You may assume \( y(x) > 0 \) for all \( x \). What happens to your solution as \( x \to \infty \)?

10. Consider the autonomous differential equation \( y' = y(y - k) \). Find all values of \( k \) such that

   a. \( y = k \) is a stable equilibrium.
   b. \( y = k \) is an unstable equilibrium.
   c. \( y = k \) is a metastable equilibrium.
   d. \( y = k \) and \( y = 0 \) are both stable equilibria.

   Or state that no such values exist.

11. A fish population in Lake Miktak is governed by the differential equation

\[
P' = 0.1P(4000 - P)
\]

If the initial population of fish is \( > 0 \), then approximately how many fish do you expect there to be in the lake after many years pass? (Ignoring silly non-mathematical facts/questions like “a single fish cannot reproduce on its own” or “this model ignores the seasonal nature of fish mating” or “what if there is some kind of environmental catastrophe?”)
III. Parametric Curves and Arclength

13. Describe the curve traced by
   
a. \((t \cos t, -2t \cos t)\) for \(0 \leq t \leq \pi\).
   
b. \((\cos(2t), \sin(2t))\) for \(0 \leq t \leq \pi\).
   
c. \((t^2, 2t + 1)\) for \(0 \leq t \leq 2\)

14. If the motion of a particle is described by the position function \((t^3 - 9t^2 + 24t, t + \frac{4}{t})\) for \(1 \leq t \leq 4\)
   
a. Does the particle ever stop?
   
b. Does the curve drawn by the particle ever have horizontal tangent slope?
   
c. Does the curve drawn by the particle ever have vertical tangent slope?

15. Consider the segment of the implicit curve \(4x + y^2 = 4\) from \((x, y) = (0, 2)\) to \(x = (1, 0)\). What is the length of this curve? You may use an integration table if necessary.