Lecture 11: Integrals for Area and Volume

Let $R$ be a region in the plane (bounded by curves).

The area of $R = \int_a^b h(x) \, dx$ where
- $a$ and $b$ are the min/max x-coords
- $h(x)$ is the height of $R$ at $x$

Ex: Let $R$ be the region bounded above by $y = 2-x^2$ and below by $y = x$. Find the area of $R$.

Find $a, b$ by solving:

$2-x^2 = x \rightarrow x^2 + x - 2 = 0$
$\Rightarrow x = \frac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \frac{-1 \pm 3}{2} = -2$ or $1$

$h(x) = (2-x^2) - x$ on $-2 \leq x \leq 1$

So $\text{Area}(R) = \int_{-2}^{1} [2-x^2 - x] \, dx$

$= \left[ 2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^{1}$

$= \frac{7}{2}$

Ex: Region bounded by $y = \ln(x)$, $y = 0$, $y = 1$, and $y = 2x$.

In terms of $x$: need to find coordinates of two other points.

Note: Can also integrate w.r.t. $y$!

Area $(R) = \int_{a}^{b} w(y) \, dy$ where $w(y)$

width of cross section at $y$.

So:
- $a = 0$, $b = 1$, $w(y)$?
- $\int_{0}^{1} [e^y - \frac{y}{2}] \, dy$

$righthand - lefthand$

$(y=\ln x) - (y=2x) = [e^y - \frac{y^2}{2}]_{0}^{1}$

$(x=e^{y}) - (x=\frac{y}{2}) = \left[ e - \frac{5}{4} \right]_0^1$

$= e - \frac{5}{4}$
Volumes of solids (slicing)

Volume (S)

\[ V = \int_a^b A(x) \, dx \]

where \( A(x) \) is the cross-sectional area (lying on its side for easier visual) at \( x \)

\[ A(y) = \text{Area of a square whose width is } 2 - \frac{3}{y} \]

\[ = \left[ 2 - \frac{3}{\sqrt{y}} \right]^2 \]

\[ = 4 - 4y^{1/3} + y^{2/3} \]

\[ \int_0^8 \left[ 4 - 4y^{1/3} + y^{2/3} \right] \, dy \]

\[ = \left[ 4y - \frac{4y^{4/3}}{4/3} + \frac{y^{5/3}}{5/3} \right]_0^8 \]

\[ = \frac{16}{5} \]