Homework assignment 3, due by Friday, April 19th, at 10:00 AM

Information and instructions

The content of this assignment corresponds roughly to Lectures 7–9, 11, and 12 of the course.

- Problems 1–5 cover and extend the techniques of integration we have learned: substitution (7.1), parts (7.2), and table (7.3).
- Problems 6 and 7 are about the Second Fundamental Theorem of Calculus (6.4).
- Problems 8–10 are about using integrals to compute areas and volumes (8.1).

Read sections 6.4 and 8.1 and complete all problems below.

Problems

1. Evaluate the following integrals (using substitution, parts, or a combination):

   a. \( \int x e^x \, dx \)  
   b. \( \int x e^{x^2} \, dx \)  
   c. \( \int x^2 e^x \, dx \)  
   d. \( \int x^3 e^{x^2} \, dx \)

   Show all work and explain clearly how you arrived at your answers.

2. One integration rabbit-hole which we have not explored is the wonderful world of trigonometric integrals. You of course know \( \int \cos \theta \, d\theta = \sin \theta + C \) and \( \int \sin \theta \, d\theta = -\cos \theta + C \).

   a. At some point we also saw \( \int \tan \theta \, d\theta = \ln |\sec \theta| + C \). This formula is obtained by substitution, log, and trig rules:

      \[
      \int \tan \theta \, d\theta = \int \frac{\sin \theta}{\cos \theta} \, d\theta = -\ln |\cos \theta| + C = \ln |\sec \theta| + C
      \]

      What substitution was made (what were \( u \) and \( du \))? How is the last step justified?

   b. The very last Level 1 trigonometric integral is \( \int \sec \theta \, d\theta = \ln |\tan \theta + \sec \theta| + C \). To verify this formula, we take the derivative of the right hand side, which is

      \[
      \frac{(\tan \theta + \sec \theta)'}{\tan \theta + \sec \theta} = \frac{(\sec \theta)^2 + \tan \theta \sec \theta}{\tan \theta + \sec \theta}
      \]

      finish the verification by multiplying the last fraction above by \( \frac{(\cos \theta)^2}{(\cos \theta)^2} \) and then simplifying carefully. After the dust settles, only \( \sec \theta \) should remain.

3. In the Level 2 trigonometric integrals, the Level 1 integrands are squared:

   i. \( \int (\sin \theta)^2 \, d\theta \)  
   ii. \( \int (\cos \theta)^2 \, d\theta \)  
   iii. \( \int (\tan \theta)^2 \, d\theta \)  
   iv. \( \int (\sec \theta)^2 \, d\theta \)

   a. This time, integral (iv) is the easiest (if you remember your trig derivatives)! Evaluate it.
Integrals (i–iii) require remembering the Pythagorean trigonometric identities:

\[(\sin \theta)^2 + (\cos \theta)^2 = 1 \quad \text{and} \quad (\tan \theta)^2 + 1 = (\sec \theta)^2\]

b. Verify (by differentiation) that \(\int (\sin \theta)^2 \, d\theta = \frac{1}{2}(\theta - \sin \theta \cos \theta) + C\).

c. Use (b) and a Pythagorean identity to evaluate \(\int (\cos \theta)^2 \, d\theta\).

d. Use (a) and a Pythagorean identity to evaluate \(\int (\tan \theta)^2 \, d\theta\).

4. **Level 3** trigonometric integrals (which arise frequently in physical/engineering applications) are those of the form

\[\int (\sin \theta)^m (\cos \theta)^n \, d\theta \quad \text{and} \quad \int (\tan \theta)^m (\sec \theta)^n \, d\theta\]

where \(m\) and \(n\) are integers \(\geq 0\).

a. Remembering that \((\cos \theta)^2 = 1 - (\sin \theta)^2\), evaluate \(\int (\sin \theta)^8 (\cos \theta)^3 \, d\theta\).

b. Remembering that \((\sec \theta)^2 = (\tan \theta)^2 - 1\) and \((\tan \theta)' = (\sec \theta)^2\), evaluate \(\int (\sec \theta)^6 \, d\theta\).

c. The method you used in (b) works generally for integrating even powers of secant. Odd powers, on the other hand, are a pain! We will demonstrate with \(\int (\sec \theta)^3 \, d\theta\) because we will need it later on. Use integration by parts (and a Pythagorean identity) to prove that

\[\int (\sec \theta)^3 \, d\theta = \tan \theta \sec \theta - \int (\sec \theta)^3 \, d\theta + \int \sec \theta \, d\theta\]

*Hint: The trick, of course, is in picking your parts correctly. Which power of secant is easiest to integrate? Use that for \(dv\).*

d. Show how to use the above (and previous bits of this homework) to conclude that

\[\int (\sec \theta)^3 \, d\theta = \frac{1}{2} \tan \theta \sec \theta + \frac{1}{2} \ln |\tan \theta + \sec \theta| + C\]

5. In class, I mentioned that the length of the path along \(y = x^2\) from \(x = a\) to \(x = b\) is equal to

\[\int_a^b \sqrt{4x^2 + 1} \, dx\]

The integral above requires the use of an integration table to evaluate symbolically:

\[\int \sqrt{u^2 + a^2} \, du = \frac{1}{2} u \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 + a^2}| + C\]

Q: Where does this bizarre formula come from? A: A sorcerous integration technique called *trigonometric substitution*. We will not be learning the general method, nor will you have to reproduce this kind of argument on an exam, but it is important for you to see it at least once!

a. We will consider, for simplicity, just the integral \(\int \sqrt{y^2 + 1} \, dy\).

*(The general formula with \(u\) and \(a\) can be derived from this one by algebra and substitution.)*

Make the (strange, but powerful!) substitution \(y = \tan \theta\) and simplify, concluding that

\[\int \sqrt{y^2 + 1} \, dy = \int (\sec \theta)^3 \, d\theta \quad (!)\]
b. Combining (a) with a previous problem, we therefore have

\[
\int \sqrt{y^2 + 1} \, dy = \frac{1}{2} \tan \theta \sec \theta + \frac{1}{2} \ln |\tan \theta + \sec \theta| + C
\]

and it only remains to reexpress the answer in terms of \( y \). Of course, \( \tan \theta = y \) (since that was the original “trigonometric substitution” we made). To finish up, solve \((\tan \theta)^2 + 1 = (\sec \theta)^2\) for \( \sec \theta \) in terms of \( y \), and then use it to complete the evaluation of \( \int \sqrt{y^2 + 1} \, dy \).

6. Re-do problem 5(e) from Midterm 1 (available on the course website):
   
a. Using the Second Fundamental Theorem of Calculus; and  
b. By explicitly computing the constant \( F(x) - G(x) \).

7. Below is graphed \( y = f(t) \) in the \( ty \)-plane.

Let \( F(x) = \int_0^x f(t) \, dt \). Sketch a graph of \( y = F(x) \) in the \( xy \)-plane, labeling at least four points on the graph with their \( xy \)-coordinates.

8. Let \( R \) be the region of the \( xy \)-plane bounded above by \( y = -x^2 + 2x \) and below by \( y = x^2 - 5x + 3 \). Graph the boundary functions, shade \( R \), and compute the area of \( R \).

9. Let \( R \) be the region of the \( xy \)-plane bounded above and on the left by \( y = 2\sqrt{x} \), below by \( y = 1 \), and on the right and below by \( y = 2 - \frac{x}{5} \). Graph the boundary functions and shade \( R \).
   
a. Set up (but do not evaluate) a definite integral (or sum of definite integrals) with respect to \( x \) that expresses the area of \( R \).
   
b. Set up (but do not evaluate) a definite integral (or sum of definite integrals) with respect to \( y \) that expresses the area of \( R \).
   
c. Compute the area of \( R \) (using either integral above).

10. Let \( R \) be the region bounded on the left by \( x = 0 \), on the right by \( x = \pi \), above by \( y = 1 - \cos x \), and below by \( y = 1 + \cos x \). Let \( S \) be the solid whose base is the region \( R \), and whose cross sections perpendicular to the \( x \)-axis are semicircles (with their diameters along the base). [If it helps you visualize, \( S \) will look something like a “Hershey’s Kiss” cut down the middle, lying with its tip at the origin.]
   
a. Express the volume of \( S \) as an integral (with respect to \( x \)).
   
b. Compute the volume of \( S \).